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To Thomas Stewart Wesner

Dad

To Margot

Phil

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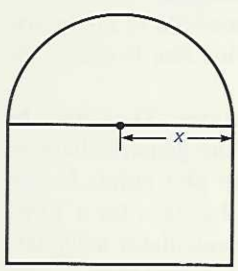
C H A P T E R

4 Polynomial and Rational Functions, and the Algebra of Functions

In this chapter we discuss two classes of functions: polynomial and rational functions. We also discuss the algebra of functions. We begin with functions whose defining expression is quadratic. The graphs of these functions are parabolas.

4–1 *Quadratic functions and functions defined on intervals*

A garden is being laid out as a semicircle attached to a rectangle. The perimeter is to be surrounded by a chain, of which 500 feet is available. What should be the dimension *x* so that the area of the garden will be maximized, and what is this area?



The solution to this problem involves quadratic functions, with which we begin this section.

Quadratic functions—parabolas

If a function is defined by a polynomial of one variable that is quadratic (degree 2) we call that function a quadratic function.

Quadratic function

A quadratic function is a function defined as

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

We graph such a function by plotting all the points (x,y) where $y = f(x)$. In other words, we first rewrite the function as $y = ax^2 + bx + c$. The graphs of these functions are parabolas. The path of a football thrown through the air and the path of a space vehicle that has exactly enough velocity to escape the attraction of the earth are both parabolic. A function that might be used to maximize the profit of a company in certain circumstances is a quadratic function, so graphing this function produces a parabola.

In section 3–4 we used the idea of horizontal and vertical translations and vertical scaling to graph certain quadratic functions. These quadratic functions were in a form in which we could apply these ideas. We call this form the vertex form for a quadratic function.

Vertex form for a quadratic function

The graph of

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

has vertex at (h,k) and is the graph of $y = x^2$, vertically scaled by a units. If $a < 0$ the parabola opens downward, and if $a > 0$ the parabola opens upward.

The vertex is at (h,k) because there is a horizontal translation of h units and a vertical translation of k units.

Figure 4–1 summarizes the features of a parabola that we use in this section: x-intercepts, y-intercept, vertex.

Since the vertex of $y = x^2$ is at (0,0), the location of the vertex of a general parabola is determined by the amount of horizontal and vertical translation of the point (0,0). This is illustrated in the examples.

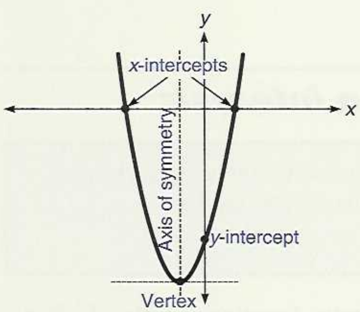


Figure 4–1

When a quadratic function is not in vertex form, it can be put in that form by completing the square on the variable x. Rewriting the equation in this form allows us to see it as a translated, scaled version of $y = x^2$.

Complete the square as in section 3–5, by taking half the value of the coefficient of the x-term (the linear term) and squaring it. However, in this case, do not add this value to both members of the equation—instead add it and its negative to the same member of the equation. This is equivalent to adding zero, which has no effect on the value of the expression.

When graphing, it is also helpful to note that *the parabola is symmetric about the vertical line that passes through the vertex.* This line is called the **axis of symmetry.**

Example 4–1 A illustrates graphing quadratic functions. They may be graphed by plotting the intercepts and vertex and using the general shape of a parabola, or, of course, by letting a graphing calculator plot points for us. The steps for hand sketching are shown here as well as the steps for a TI-81 graphing calculator, along with a suggested range for the calculator's display. When using a graphing calculator we can graph the function and then use the trace and zoom capabilities to find approximate values of the vertex.

■ *Example 4–1 A*

Graph the parabola after putting the function in vertex form. Compute the vertex and intercepts.

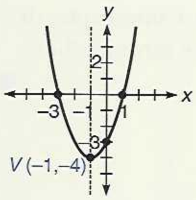
1. $f(x) = x^2 + 2x - 3$

Complete the square on x:

$y = x^2 + 2x + 1 - 1 - 3$ — Half of 2 is 1, and $1^2 = 1$; Add +1 and −1 to the right member

$y = (x + 1)^2 - 4$ — $x^2 + 2x + 1 = (x + 1)^2$

$y = (x - (-1))^2 - 4$ — Vertex form



[Y=] [X|T] [x^2] [+] 2 [X|T] [−] 3
[RANGE −4,2,−5,2]

Vertex: $(-1,-4)$

y-intercept:

$$y = 0^2 + 2(0) - 3$$ Let $x = 0$ in $y = x^2 + 2x - 3$

$$y = -3$$ y-intercept

x-intercept:

$$0 = x^2 + 2x - 3$$ Let $y = 0$ in $y = x^2 + 2x - 3$

$$0 = (x + 3)(x - 1)$$ Factor

$$x + 3 = 0 \text{ or } x - 1 = 0$$ Zero product property

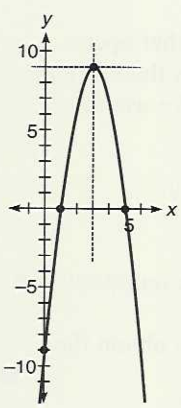
$$x = -3 \text{ or } x = 1$$ x-intercepts

Since the coefficient of x^2, 1, is positive, the parabola opens upward. To find approximate values of the vertex in this problem with the TI-81 we graph it with the calculator, as shown, and then proceed as follows.

[TRACE] The cursor appears on the graph. Move it as close to the vertex as possible.

[ZOOM] 2 This expands the graph.

Repeat this trace and zoom process several times. Each time we do this we find a more and more accurate value. The most accurate values are found by algebraic processes, such as by completing the square. In practical cases where this must be done for many quadratic equations we could develop a formula that would give exact results. (See problem 58 in the exercises.) Here, however, we are as interested in getting experience with this type of function as we are in obtaining numeric results. When $a \neq 1$ we must first factor a from the expression to complete the square.



[Y=] [(−)] 2 [X|T] [x^2] [+] 1 2 [X|T] [−] 9
[RANGE −1,7,−10,10]

2. $f(x) = -2x^2 + 12x - 9$

Complete the square:

$$y = -2(x^2 - 6x) - 9$$ Since $a \neq 1$ we factor it from the x-terms

$$y = -2(x^2 - 6x + 9) + 2(9) - 9$$ We added $-2(9)$, so we must add $+2(9)$

$$y = -2(x - 3)^2 + 9$$ Vertex form

Vertex: $(3,9)$

y-intercept

$$y = -2(0)^2 + 12(0) - 9 = -9$$ Let $x = 0$ in $-2x^2 + 12x - 9$

x-intercept

$$0 = -2x^2 + 12x - 9$$ Let $y = 0$ in $y = -2x^2 + 12x - 9$

$$0 = 2x^2 - 12x + 9$$

$$x = \frac{6 \pm 3\sqrt{2}}{2} \approx 0.9, 5.1$$ Quadratic formula

Since $a < 0$ the parabola opens downward. Plot the two x-intercepts, the y-intercept, and the vertex. Draw the parabola that passes through these points. ■

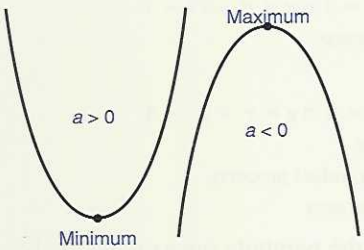


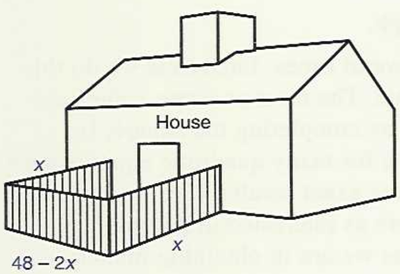
Figure 4–2

Maximum and minimum values of quadratic functions

The vertex (h,k) of a parabola $y = ax^2 + bx + c$ is the lowest point on the graph when $a > 0$, and the highest point when $a < 0$ (figure 4–2). The value of k at these points is said to be a *minimum* or *maximum value* at these points, and h is the value of x at which this minimum or maximum occurs. This has many applications, one of which is illustrated in example 4–1 B.

■ ***Example 4–1 B***

A homeowner has 48 feet of fencing to fence off an area behind the home. The home will serve as one boundary. What are the dimensions of the maximum area that can be fenced off, and what is the area?



As shown in the figure, let x represent the length of each of the two sides that are perpendicular to the house. Since there is 48 feet of fencing available, there is $48 - 2x$ feet remaining for the third side of the fence. The area of a rectangle is the product of its length and width, so, if A represents area, we have the equation

$$A = x(48 - 2x)$$
$$A = -2x^2 + 48x$$
$$A = -2(x^2 - 24x)$$
$$A = -2(x^2 - 24x + 144) + 2(144) \qquad \text{Complete the square}$$
$$A = -2(x - 12)^2 + 288$$

The vertex is (12,288).

Since the coefficient of x^2 is negative, this is a parabola that opens downward, so that its vertex is at its highest point, and this is therefore the maximum value of A. The x-coordinate of the vertex is 12. We use this value of x to compute A:

$$A = -2x^2 + 48x$$
$$A = -2(12^2) + 48(12)$$
$$A = 288$$

Observe that this is the y-coordinate at the vertex. This value represents the maximum value that A can take on in this situation.

The third side is $48 - 2x = 48 - 24 = 24$ feet. Thus, to obtain the maximum area of 288 ft^2 the dimensions are 12 ft by 24 ft. ■

Functions defined on intervals

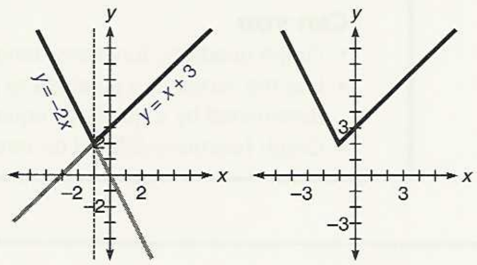
Functions are often defined with more than one expression. Each expression applies to a particular interval. Each expression is graphed as usual, but we use only the part of the graph indicated by the inequalities. This is illustrated in example 4–1 C.

Part 2 of example 4–1 C shows how to graph functions defined on intervals using the TI-81 graphing calculator.

■ ***Example 4–1 C*** Graph the following functions. State all intercepts.

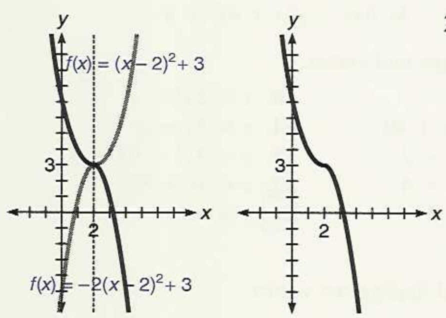
1. $f(x) = \begin{cases} -2x, & x < -1 \\ x + 3, & x \geq -1 \end{cases}$

For the interval $x < -1$ we graph the straight line $y = -2x$. For the interval $x \geq -1$ we graph the straight line $y = x + 3$, which gives a y-intercept at $y = 3$. The figure shows this process.



$f(x) = \begin{cases} -2x, & x < -1 \\ x + 3, & x \geq -1 \end{cases}$

2. $f(x) = \begin{cases} (x-2)^2 + 3, & x < 2 \\ -2(x-2)^2 + 3, & x \geq 2 \end{cases}$



Interval: $x < 2$

$y = (x-2)^2 + 3$; parabola, opens upward, vertex at (2,3), y-intercept at (0,7).

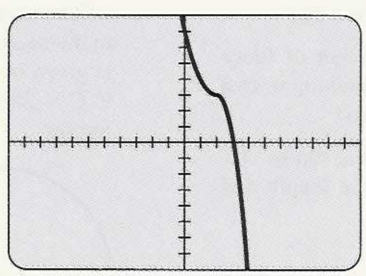
Interval: $x \geq 2$

$y = -2(x-2)^2 + 3$; parabola, opens downward, vertex at (2,3), y-intercept at (0,−5).

On the TI-81 a ''TEST'' evaluates to zero if false and one if true. Therefore, to graph this function we enter the following for Y_1 (using the $\boxed{\text{Y=}}$ key). Use the standard range setting.

$:Y_1 = ((X-2)^2 + 3)(X<2) + (-2(X-2)^2 + 3)(X\geq 2)$

The $<$ and $\geq$ symbols are found using the TEST window ($\boxed{\text{2nd}}$ $\boxed{\text{MATH}}$)



The idea is that if $x < 2$ then Y_1 is equivalent to $((x-2)^2+3)(1) + (-2(x-2)^2+3)(0) = (x-2)^2+3$. Similarly, when $x \geq 2$ Y_1 is equivalent to $-2(x-2)^2+3$.

Mastery points

Can you

- Graph quadratic functions using the vertex and intercepts?
- Use the vertex of a parabola to maximize or minimize a value determined by a quadratic equation?
- Graph functions defined on intervals?

Exercise 4–1

Graph the following parabolas. State the intercepts and vertex.

1. $f(x) = (x-1)^2+3$
2. $f(x) = (x+2)^2-1$
3. $f(x) = 2(x+3)^2-4$
4. $f(x) = -2(x-1)^2-6$
5. $f(x) = -(x-5)^2-1$
6. $f(x) = \frac{1}{2}(x+4)^2+8$

Graph the following parabolas, completing the square if necessary. State the intercepts and vertex.

7. $y = x^2 - x - 6$
8. $y = x^2 + 2x - 15$
9. $y = 3x^2$
10. $y = 2x^2 - x$
11. $y = x^2 + 3x$
12. $y = \frac{1}{2}x^2$
13. $y = -x^2 + 3x + 40$
14. $y = 5x - x^2$
15. $y = x^2 - 4$
16. $y = 9 - x^2$
17. $y = 3x^2 + 6x - 2$
18. $y = 3x^2 - 9x + 8$
19. $y = x^2 - 5x - 8$
20. $y = 4 - x - x^2$
21. $y = 2x^2 - 4x - 4$
22. $y = x^2 + 3x - 3$
23. $y = x^2 + 4$
24. $y = x^2 + x - 3$
25. $y = x^2 - x + 5$
26. $y = -2x^2 - 2x + 1$
27. $y = -x^2 - x$
28. $y = 3x^2 + 9x$

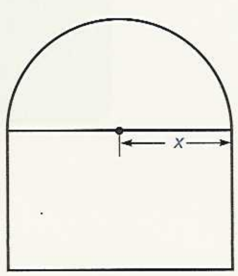
Solve the following problems by creating an appropriate second-degree equation and finding the vertex.

29. A homeowner has 260 feet of fencing to fence off a rectangular area behind the home. The home will serve as one boundary (the fence is only necessary for three of the four sides). What are the dimensions of the maximum area that can be fenced off, and what is the area?

30. The homeowner of problem 29 has 300 feet of fence available. What are the dimensions of the maximum area that can be fenced off, and what is the area?

31. What is the area of the largest rectangle that can be created with 260 feet of fence? What are the length and width of this rectangle?

32. A garden is being laid out as a semicircle attached to a rectangle (see the figure). The perimeter is to be surrounded by a chain, of which 500 feet is available. What should be the dimension x so that the area of the garden will be maximized, to the nearest foot? What is this area, to the nearest square foot? Recall that the area of a circle is given by $A = \pi r^2$, and the perimeter (circumference) is $C = 2\pi r$, where r is the radius of the circle (x in the figure).



33. If an object is thrown into the air with an initial vertical velocity of v_0 ft/s, then its distance above the ground s, for time t, is given by $s = v_0t - 16t^2$. Suppose an object is thrown upward with initial velocity 64 ft/s; find how high the object will go (when s is a maximum) and when it will return to the ground (when $s = 0$).

34. An arrow is shot into the air with an initial vertical velocity of 48 ft/s. Find out how high the arrow will go, and when it will return to the ground. See problem 33.

35. Suppose the velocity distribution of natural gas flowing smoothly in a certain pipeline is given by $V = 6x - x^2$, where V is the velocity in meters per second and x is the distance in meters from the inside wall of the pipe. What is the maximum velocity of the gas, and where does this occur?

36. The power output P (in watts), of an automobile alternator that generates 14 volts and has an internal resistance of 0.20 ohms is given by $P = 14I - 0.20I^2$. At what current I (in amperes) does the generator generate maximum power, and what is the maximum power?

37. If a company's profit P (in dollars) for a given week when producing u units of a commodity in that week is $P = -u^2 + 100u - 1{,}000$, how many units must be made to produce the maximum profit, and what is this profit?

38. Find the two numbers whose sum is 100 and whose product is a maximum.

39. Given that the difference between two numbers is 8, what is the minimum product of the two numbers? Also, what are the numbers?

40. Find the minimum product of two numbers whose difference is 12; what are the numbers?

41. For a given perimeter, will a circle or rectangle contain a greater area? To find out, assume a length P for the total perimeter, and maximize the area for a rectangle with perimeter P (as in problem 31), then compute the area of a circle with perimeter (circumference) P.

42. The method of completing the square can be used to derive the quadratic formula. This states:

If $ax^2 + bx + c = 0$ and $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Derive the formula by assuming that $ax^2 + bx + c = 0$; then add $-c$ to both sides and complete the square on x.

43. An alternate method of deriving the quadratic formula (see problem 42) was devised by the Hindus. About the year 1025, the Indian Sridhara presented the following method.[1]

- **a.** Multiply each member of $ax^2 + bx + c = 0$ by $4a$.
- **b.** Add b^2 to both members.
- **c.** Subtract $4ac$ from both members.
- **d.** Observe that the left member is now a perfect square, then take the square root of both members.

Use this method to derive the quadratic formula.

[1]From K. R. S. Sastry, "The Quadratic Formula: A Historic Approach," *The Mathematics Teacher*, November 1988.

Graph the following functions defined on intervals.

44. $f(x) = \begin{cases} -2x + 1, & x \geq -1 \\ 3x + 1, & x < -1 \end{cases}$

45. $g(x) = \begin{cases} -\frac{1}{2}x, & x < 0 \\ -3x, & x \geq 0 \end{cases}$

46. $h(x) = \begin{cases} x + 1, & x < -1 \\ \frac{1}{2}x + \frac{1}{2}, & x \geq -1 \end{cases}$

47. $f(x) = \begin{cases} 2x - 1, & x < \frac{1}{2} \\ \frac{2}{3}x - \frac{1}{3}, & x \geq \frac{1}{2} \end{cases}$

48. $g(x) = \begin{cases} x^2 - 4x + 4, & x < 2 \\ -\frac{3}{4}x + \frac{3}{2}, & x \geq 2 \end{cases}$

49. $h(x) = \begin{cases} x^2 - 2x + 4, & x < 1 \\ -2x + 5, & x \geq 1 \end{cases}$

50. $f(x) = \begin{cases} x^2 + 2, & x < 0 \\ -2x^2 + 2, & x \geq 0 \end{cases}$

51. $g(x) = \begin{cases} -2x^2 + 8x - 10, & x < 2 \\ x^2 - 4x + 2, & x \geq 2 \end{cases}$

52. $g(x) = \begin{cases} -x, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$

53. $g(x) = \begin{cases} x, & x < 0 \\ x^3, & x \geq 0 \end{cases}$

54. $g(x) = \begin{cases} x^3, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$

55. $g(x) = \begin{cases} x^2, & x < 0 \\ -\sqrt{x}, & x \geq 0 \end{cases}$

56. $g(x) = \begin{cases} x, & x < 1 \\ x^3 + 1, & x \geq 1 \end{cases}$

57. $g(x) = \begin{cases} x - 1, & x < 1 \\ \sqrt{x - 1}, & x \geq 1 \end{cases}$

58. There is a formula for each of the coordinates of the vertex of the general parabola when the equation is given in the form $f(x) = ax^2 + bx + c$. This formula can be found by completing the square using the literal constants a, b, and c instead of numeric values. Find the formula for the x and the y coordinates of the vertex.

Skill and review

1. Factor $3x^2 + x - 10$.
2. Factor $3x^2 + 13x - 10$.
3. Factor $x^4 - 16$.
4. List all the prime divisors of 96.
5. List all the positive integer divisors of 96.
6. If $f(x) = 2x^3 - x^2 - 6x + 20$, find $f(-2)$.
7. Use long division to divide $2x^3 - x^2 - 6x + 20$ by $x^2 + 2$.
8. Graph $f(x) = \sqrt{x - 2} - 3$.
9. Compute $f \circ g(x)$ and $g \circ f(x)$ if $f(x) = x^4 - 6x^2 + 8$ and $g(x) = \sqrt{x + 1}$.

4–2 Polynomial functions and synthetic division

Napthalene ($C_{10}H_8$) is a very stable chemical.[2] To determine its stability one needs to solve the equation $x^{10} - 11x^8 + 41x^6 - 65x^4 + 43x^2 - 9 = 0$. It has 10 solutions. Find them to four decimal places of accuracy.

Solving equations like the one in the section opening problem is very common in science, engineering, business and finance, and anywhere mathematics is applied. Most sections of this book include problems that require solving some type of equation. In this and the next few sections, we investigate this important part of mathematics. The problem concerning napthalene appears in exercise set 4–3. This section presents algebraic methods for solving polynomial equations. Section 4–3 presents ways to graph polynomial functions and ways to solve equations with graphical calculators.

Some terminology

A zero of a function f is a value c such that $f(c) = 0$. If the function is defined by a polynomial, this means the polynomial evaluates to zero when x is replaced by c. In this section, we examine algebraic and graphical methods for finding zeros of functions whose defining rule is a polynomial in one variable with real coefficients. The implied domain of such functions is the set of real numbers. *Although most of what we discuss in this section extends to include the complex number system we restrict our discussion to real numbers.*

We begin with a definition of polynomial functions.

Polynomial function

A polynomial function in one variable is a function of the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,\ a_n \neq 0$$

where $a_0, a_1, a_2, \ldots, a_n$ are constant, real-valued coefficients.

[2] From Professor Jun-Ichi Aihara, "Why Aromatic Compounds Are Stable," *Scientific American,* March 1992.

The coefficient a_n is called the **leading coefficient,** and n is the **degree of the polynomial.** The term a_0 is the **constant term.** When the polynomial is written as above, with exponents in descending order, we say it is written in the **standard form** for a polynomial.

A linear function is a polynomial function of degree 0 or 1, and a quadratic function is a polynomial function of degree 2. A polynomial function of degree 0 is also called a **constant function.** By way of example:

Function	Degree	
$f(x) = 4$	0	Constant function
$f(x) = -3x + 4$	1	Linear function
$f(x) = 5x^2 - 3x + 4$	2	Quadratic function
$f(x) = x^3 + 5x^2 - 3x + 4$	3	Polynomial function of degree 3

An important property of a function is those values of the domain for which $f(x)$ is zero. As noted above these values are called zeros of a function. When we graph these functions in section 4–3, we will see that *a zero of a function corresponds to an x-intercept of the graph of the function.*

Zeros of a function

If $f(x)$ defines a function, and for some real number c in the domain of f, $f(c) = 0$, then c is called a zero of the function.

Zeros of polynomial functions of degree 0, 1, and 2

We have algebraic methods to find zeros of linear and quadratic polynomial functions (functions of degrees 0, 1, and 2). We *replace f(x) by 0* and solve.[3] Methods for solving linear and quadratic equations were covered in chapter 2.

Zeros of polynomial functions of degree 3 and above

We begin our discussion of algebraic methods for finding zeros of functions by stating several facts, without proof, that can be of help.

Maximum number of zeros

A polynomial function of degree n has at most n zeros.

Concept

For a given polynomial function $f(x)$ of degree n there are at most n values $c_1, c_2, \ldots, c_n$ such that $f(c_1) = 0, f(c_2) = 0, \ldots, f(c_n) = 0$.

This tells us that *the graph of a polynomial function of degree n has at most n x-intercepts.*

[3]The Babylonians of 4,000 years ago could deal with all three of these situations.

Another important fact is that it is possible to find all *rational* zeros of polynomial functions.

Rational zero theorem

If $\frac{p}{q}$ is a rational number in lowest terms and is a zero of a polynomial function, then the denominator q is a factor of the leading coefficient, a_n, and the numerator p is a factor of the constant term, a_0.

As a memory aid remember that any rational zero is of the form

$$\frac{\text{Factor of } a_0}{\text{Factor of } a_n}$$

Example 4–2 A illustrates the use of the rational zero theorem.

Example 4–2 A

List all possible rational zeros for the polynomial function $f(x) = 6x^4 + 25x^3 - 15x^2 - 25x + 9$.

According to the theorem above, if $f\left(\frac{p}{q}\right) = 0$ then p, the numerator, divides 9 and q, the denominator, divides 6. Thus, we need to check all positive and negative fractions where the numerator is 1, 3, or 9, and the denominator is 1, 2, 3, or 6. We obtain these values as shown.

Numerator is 1: $\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$

Numerator is 3: $\pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{3}{3}, \pm\frac{3}{6}$

Numerator is 9: $\pm\frac{9}{1}, \pm\frac{9}{2}, \pm\frac{9}{3}, \pm\frac{9}{6}$

After reducing, the set of possible rational zeros of f is

$$\pm 1, \pm 3, \pm 9, \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{9}{2}, \pm\tfrac{1}{3}, \pm\tfrac{1}{6}$$

The remainder theorem

To find out which possible rational zeros are actually zeros we will use a method called synthetic division; to understand it we must first consider the following theorem.

Remainder theorem

If f is a nonconstant polynomial function and c is a real number, then the remainder when $x - c$ is divided into $f(x)$ is $f(c)$.

This implies that c is a zero of f *if and only if there is no remainder* when $f(x)$ is divided by $x - c$. We could use long division to test values as potential zeros. Rather than go back to long division, however, we introduce a faster algorithm called **synthetic division.** This algorithm is *equivalent to long division by a linear factor $x - c$.*

Synthetic division

We illustrate synthetic division with the example:

$$f(x) = x^3 - 2x^2 - 5x - 6 \quad \text{and} \quad c = 4$$

Computation would show that $f(4) = 6$. The remainder theorem states that this can also be found by computing $(x^3 - 2x^2 - 5x - 6) \div (x - 4)$.

We construct a table using the coefficients of the polynomial dividend and the value of c at the left.

	1	−2	−5	−6
4				

We proceed with the first column on the left, moving column by column to the right, *performing the same steps on each column:*

1. Add the values in the first two rows; write the result in the third row.
2. Multiply the third row by the value c and put the result in the second row of the next column.

The first column will have only one value initially, so we do not need to perform any addition in that column—we simply bring down the value.

	1	−2	−5	−6	Bring down the 1
		4			Multiply by 4
4	1				

	1	−2	−5	−6	Add 4 + (−2)
		4	8		Multiply the result by 4
4	1	2			

	1	−2	−5	−6	Add 8 + (−5)
		4	8	12	Multiply the result by 4
4	1	2	3		

	1	−2	−5	−6	Add (−6) + 12
		4	8	12	
4	1	2	3	$\boxed{6}$	

Looking at the last table we see the value $f(4) = 6$ appear in the last computation.

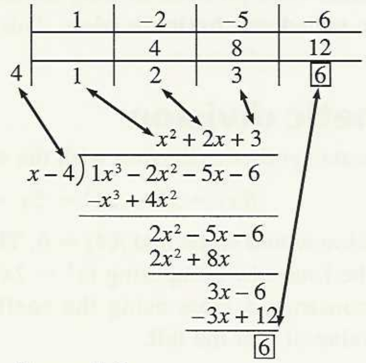


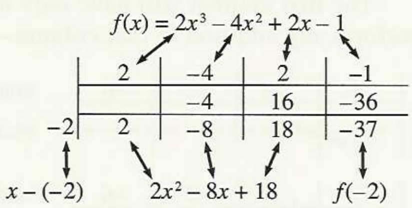
Figure 4–3

Figure 4–3 shows the similarities between long division and synthetic division. (In the long division we changed the signs of the terms that are subtracted.) Observe also that the coefficients of the quotient $x^2 + 2x + 3$ are in the synthetic division table.

Example 4–2 B shows the use of synthetic division to evaluate a function and to perform algebraic division by a linear factor.

■ Example 4–2 B

1. $f(x) = 2x^3 - 4x^2 + 2x - 1$; (a) find $f(-2)$, (b) divide f by $x + 2$.



The bottom line of this table gives two results. It tells us that $f(-2)$ is -37, which is the answer to (a). It also tells us what would happen if we divided by $x - c$, or $x - (-2) = x + 2$, which is the answer to (b). It tells us that $\dfrac{2x^3 - 4x^2 + 2x - 1}{x + 2} = 2x^2 - 8x + 18$, with remainder -37. We know that the result is of degree 2 since we have divided a polynomial of degree 3 by a divisor of degree 1, leaving a quotient of degree $3 - 1 = 2$.

2. Divide $6x^2 + x - 2$ by $x - \frac{1}{2}$.

	6	1	−2
		3	2
$\frac{1}{2}$	6	4	0

The first two values of the last line produce a first-degree polynomial (a linear expression), which is $6x + 4$. The 0 is the remainder. Hence we know that $\dfrac{6x^2 + x - 2}{x - \frac{1}{2}} = 6x + 4$, with no remainder.

This last result means that $x - \frac{1}{2}$ is a factor of $6x^2 + x - 2$. In particular, $6x^2 + x - 2 = (x - \frac{1}{2})(6x + 4)$.

3. Divide $x^5 - 3x^3 + x$ by $x - 2$.

We first rewrite this expression as $x^5 + 0x^4 - 3x^3 + 0x^2 + 1x + 0$ to obtain the coefficients for the table: 1, 0, −3, 0, 1, and 0.

	1	0	−3	0	1	0
		2	4	2	4	10
2	1	2	1	2	5	10

This last line gives the result $\dfrac{x^5 - 3x^3 + x}{x - 2} = x^4 + 2x^3 + x^2 + 2x + 5$ with remainder 10.

Note This result also means that if we evaluate $x^5 - 3x^3 + x$ for 2 we obtain the value 10. That is, $2^5 - 3(2^3) + 2 = 10$.

Zeros and factoring

Whenever we get a remainder of zero using synthetic division with $x - c$, we have found a zero, c, and a factor of $f(x)$, namely $x - c$. Thus, we can use zeros to help factor a polynomial, and we can use factoring to find zeros.

Example 4–2 C illustrates using synthetic division to find linear factors and to thereby factor a polynomial expression.

Example 4–2 C

Find all rational zeros of the given function. If possible find all real zeros. Also, factor using rational values.

1. $f(x) = 6x^4 - 5x^3 - 39x^2 - 4x + 12$

Using the rational zero theorem all possible rational zeros of this function can be determined to be $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 2, \pm\frac{2}{3}, \pm 3, \pm\frac{3}{2}, \pm 4, \pm\frac{4}{3}, \pm 6$, and ± 12.

We start with 1, then −1, etc. It can be verified that neither 1 nor −1 is a zero. Therefore try $\frac{1}{2}$ next.

	6	−5	−39	−4	12
		3	−1	−20	−12
$\frac{1}{2}$	6	−2	−40	−24	0

Since the remainder is 0, this is a zero of f, and $(x - \frac{1}{2})$ is a factor of $f(x)$

$$\begin{aligned} f(x) &= (x - \tfrac{1}{2})(6x^3 - 2x^2 - 40x - 24) && \text{From the last line of the table} \\ &= 2(x - \tfrac{1}{2})(3x^3 - x^2 - 20x - 12) && \text{Common factor of 2} \end{aligned}$$

Any remaining zero of f will also be a zero of $3x^3 - x^2 - 20x - 12$, so we attack this problem with synthetic division. All possible rational zeros of this expression are $\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 3, \pm 4, \pm\frac{4}{3}, \pm 6$, and ± 12.

However, if ± 1 were zeros they would have been zeros of f, so we do not check them again. Neither $\frac{1}{3}$ nor $-\frac{1}{3}$ is a zero; neither is 2.

	3	-1	-20	-12
		-6	14	12
-2	3	-7	-6	0

$(x + 2)$ is a factor of $f(x)$

$$f(x) = 2(x - \tfrac{1}{2})(x + 2)(3x^2 - 7x - 6)$$
$$f(x) = 2(x - \tfrac{1}{2})(x + 2)(3x + 2)(x - 3)$$

Factor the quadratic trinomial

$$f(x) = 2(x - \tfrac{1}{2})(x + 2)(3)(x + \tfrac{2}{3})(x - 3)$$

$3x + 2 = 3(x + \frac{2}{3})$

$$f(x) = 6(x - \tfrac{1}{2})(x + 2)(x + \tfrac{2}{3})(x - 3)$$

Factoring $3x + 2$ in this way is not necessary, but it makes all of the linear factors of the form "$x + a$" (that is, leading coefficient one). This factorization has given the four zeros of f: $\frac{1}{2}$, -2, $-\frac{2}{3}$, and 3.

2. $f(x) = x^3 - 3x^2 - 2x + 8$

Possible rational zeros are ± 1, ± 2, ± 4, and ± 8. Checking 2 gives

	1	-3	-2	8
		2	-2	-8
2	1	-1	-4	0

Thus $f(x) = (x - 2)(x^2 - x - 4)$.

The quadratic trinomial $x^2 - x - 4$ does not factor, but its zeros can be found with the quadratic formula. If $0 = x^2 - x - 4$ then

$$x = \frac{1 \pm \sqrt{17}}{2}.$$

Thus, the real zeros of f are 2, $\frac{1 + \sqrt{17}}{2}$, $\frac{1 - \sqrt{17}}{2}$, and

$$f(x) = (x - 2)\left(x - \frac{1 + \sqrt{17}}{2}\right)\left(x - \frac{1 - \sqrt{17}}{2}\right).$$

Note See ways to factor quadratic expressions in section 2–2. Any quadratic expression can be factored using the quadratic formula as shown in that section. Thus, *we never need to use synthetic division on quadratic polynomials.*

Prime factorization of polynomials

In part 2 of example 4–2 C we saw a quadratic polynomial that factored over the real number system. When the zeros of a quadratic expression are complex we say that the expression is **prime over the real number system.** This is determined by the value of the discriminant $b^2 - 4ac$. If $b^2 - 4ac < 0$ then the quadratic is prime (over the real number system) since the formula produces complex zeros. In this case we choose not to factor the quadratic.

The next theorem comes from work done by the famous German mathematician Karl Friedrich Gauss by the year 1799.

> **Prime factorization of polynomials over the real number system**
> Every polynomial of positive degree n with real coefficients can be expressed as the product of a real number, linear factors, and prime quadratic factors.

This theorem states that if a polynomial function has only real coefficients, then it can be factored into a product of a real number and linear and quadratic factors, where the quadratic factors have only complex zeros. It is not always easy to find all these factors, but Gauss proved that it is possible in theory. In this book we examine only selected problems in which we have some chance of success.

Before illustrating Gauss's theorem, we should also discuss the idea of *multiplicity* of a zero. By way of example, if $f(x) = (x - 2)(x + 3)^2(x - \frac{1}{2})^4$, then 2 is a zero of multiplicity one, -3 a zero of multiplicity 2, and $\frac{1}{2}$ a zero of multiplicity 4. This is formalized in the following definition.

> **Multiplicity of zeros**
> If $f(x)$ is defined by a polynomial expression, c is a real number in the domain of f, and $(x - c)^n$ divides $f(x)$, but $(x - c)^{n+1}$ does not divide $f(x)$, then we say that c is a zero of multiplicity n.

Example 4–2 D illustrates Gauss's theorem about the prime factorization of polynomials over the real number system, and the idea of multiplicity of zeros.

■ **Example 4–2 D**

Given $f(x) = 4x^5 - 10x^4 + 6x^3 - 4x^2 + 10x - 6$, (a) factor the polynomial completely and (b) list the zeros of the polynomial; note if any zeros have multiplicity greater than one.

$$\begin{aligned} f(x) &= 4x^5 - 10x^4 + 6x^3 - 4x^2 + 10x - 6 \\ &= 2(2x^5 - 5x^4 + 3x^3 - 2x^2 + 5x - 3) \qquad \text{Common factor of 2} \end{aligned}$$

We now factor $2x^5 - 5x^4 + 3x^3 - 2x^2 + 5x - 3$.

Possible rational zeros are ±1, ±3, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$.

$$\begin{array}{r|rrrrrr} & 2 & -5 & 3 & -2 & 5 & -3 \\ & & 2 & -3 & 0 & -2 & 3 \\ \hline 1 & 2 & -3 & 0 & -2 & 3 & 0 \end{array} \qquad (x-1) \text{ is a factor of } f(x)$$

$$f(x) = 2(x - 1)(2x^4 - 3x^3 - 2x + 3)$$

$2x^4 - 3x^3 - 2x + 3$ has the same possible rational zeros: ± 1, ± 3, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$, and we could proceed by synthetic division. However, the pattern of the coefficients suggests we can *factor using grouping:*

$$\begin{aligned} 2x^4 - 3x^3 - 2x + 3 &= x^3(2x - 3) - 1(2x - 3) \\ &= (2x - 3)(x^3 - 1) \\ &= (2x - 3)(x - 1)(x^2 + x + 1) \\ &= 2(x - \tfrac{3}{2})(x - 1)(x^2 + x + 1) \end{aligned}$$

$x^3 - 1$ is a difference of two cubes

The zeros of $x^2 + x + 1$ are complex ($b^2 - 4ac < 0$), so we would say that we have completely factored the expression over the real numbers. Thus, $f(x) = 2(x - 1)[2(x - \frac{3}{2})(x - 1)(x^2 + x + 1)]$ and the answer to part (a) is

$$f(x) = 4(x - 1)^2(x - \tfrac{3}{2})(x^2 + x + 1)$$

The answer to part (b) is 1 and $\frac{3}{2}$, with 1 having multiplicity 2. ■

Bounds for real zeros—where to look

It is often possible to avoid testing all of the possible rational zeros that are given by the rational zero theorem. When we apply synthetic division, the last line can give an indication about bounds for real zeros.

Upper and lower bounds for real zeros

- The real number U is an **upper bound** for the real zeros of a polynomial function if U is greater than or equal to all the zeros of the function.
- The real number L is a **lower bound** for the real zeros of a polynomial function if L is less than or equal to all the zeros of the function.

The following theorem can tell us when a value is an upper or lower bound for the real zeros of a polynomial function.

Bounds theorem for real zeros

Let c be a real number and $f(x)$ be a polynomial function with real coefficients and positive leading coefficient; consider all the coefficients in the last line of the synthetic division algorithm as applied to the value c. Then c is

- an *upper bound* if $c \geq 0$ and these coefficients are all positive or zero.
- a *lower bound* if $c \leq 0$ and the signs of the values in the last row alternate, except that zero may be written as $+0$ or -0 when considering sign alternation.

There is no need to check possible rational zeros that are greater than an upper bound or that are less than a lower bound. (Note that this theorem applies to irrational as well as rational zeros.) This is illustrated in example 4–2 E.

■ ***Example 4–2 E***

Find upper and lower bounds for the real zeros of the function $f(x) = x^4 + 3x^3 + 2x^2 - 5x + 12$.

Possible rational zeros are ± 1, ± 2, ± 3, ± 4, ± 6, and ± 12.

	1	3	2	−5	12
		1	4	6	1
1	1	4	6	1	13

The value 1 is an upper bound because all the coefficients in the last row are positive or zero. Thus, there would be no reason to check the values 2, 3, 4, 6, or 12.

	1	3	2	−5	12
		−3	0	−6	33
−3	1	0	2	−11	45

The value -3 is a lower bound because the coefficients in the last row alternate signs if we write the zero in the last row as -0: $1\ -0\ 2\ -11\ 45$. Thus, there would be no reason to check the possible zeros -4, -6, or -12. ■

Number of real zeros

The next theorem can be used to obtain the number of possible positive and negative real zeros. It refers to variations in the signs of the coefficients of a polynomial. If a polynomial expression is written in standard form, ignoring terms with coefficients that are 0, a **variation in sign** is said to occur if a succeeding coefficient has a different sign than the one that preceded it (see the example 4–2 F).

Descartes' rule of signs

Let $f(x)$ be a function defined by a polynomial with real coefficients. Then

- the number of *positive* real zeros is equal to the number of variations in sign in $f(x)$ or is less than this number by a multiple of 2.
- the number of *negative* real zeros is equal to the number of variations in sign in $f(-x)$ or is less than this number by a multiple of 2.

Descartes' rule of signs is illustrated in the next example.

■ *Example 4–2 F*

a. Investigate the real zeros of the given function in terms of the number of possible positive and negative real zeros.

b. List all possible rational zeros.

c. Find all rational zeros.

d. Find irrational zeros when possible; when not possible, find bounds for the irrational zeros.

e. Write the function as a product of linear and prime quadratic factors if practical.

1. $f(x) = x^5 - 8x^4 + 11x^3 + 22x^2 - 26x - 28$

a. $f(x) = x^5 - 8x^4 + 11x^3 + 22x^2 - 26x - 28$

Three changes of sign; 1 or 3 positive zeros

$f(-x) = -x^5 - 8x^4 - 11x^3 + 22x^2 + 26x - 28$

Two changes in sign; 0 or 2 negative zeros

b. The possible rational zeros are ±1, ±2, ±7, ±14, and ±28.

	1	−8	11	22	−26	−28
		−1	9	−20	−2	28
−1	1	−9	20	2	−28	0

$(x + 1)$ is a factor of $f(x)$

$f(x) = (x + 1)(x^4 - 9x^3 + 20x^2 + 2x - 28)$

	1	−9	20	2	−28
		−1	10	−30	28
−1	1	−10	30	−28	0

$(x + 1)$ is a factor of $f(x)$ for the second time; thus $(x + 1)^2$ is a factor of $f(x)$

$f(x) = (x + 1)^2(x^3 - 10x^2 + 30x - 28)$

Checking -1 again shows it is not another zero; this is also true of $+1$.

	1	−10	30	−28
		2	−16	28
2	1	−8	14	0

$(x - 2)$ is a factor of $f(x)$

$f(x) = (x + 1)^2(x - 2)(x^2 - 8x + 14)$

The zeros of $x^2 - 8x + 14$ are $4 \pm \sqrt{2}$ (using the quadratic formula). Thus,

$$\begin{aligned} x^2 - 8x + 14 &= [x - (4 - \sqrt{2})][x - (4 + \sqrt{2})] \\ &= (x - 4 + \sqrt{2})(x - 4 - \sqrt{2}) \end{aligned}$$

c. Rational zeros: -1 (multiplicity 2), 2.

d. Irrational zeros: $4 \pm \sqrt{2}$.

e. $f(x) = (x + 1)^2(x - 2)(x - 4 + \sqrt{2})(x - 4 - \sqrt{2})$.

2. $f(x) = x^4 - 5x^3 + 4x^2 + 3x - 1$

a. $f(x) = x^4 - 5x^3 + 4x^2 + 3x - 1$

Three changes of sign so there are one or three positive real zeros

$f(-x) = x^4 + 5x^3 + 4x^2 - 3x - 1$

One change in sign so there is one negative real zero

b. The only possible rational zeros are ± 1.

	1	−5	4	3	−1
		1	−4	0	3
1	1	−4	0	3	2

We see that 1 is not a zero or an upper bound. Since there are no other positive rational zeros possible, there is no sense looking for more positive integer zeros. However, we do need an upper bound.

Looking at the first two columns above shows that nothing less than 5 will serve as an integer upper bound, since anything less than 5 will cause a negative value in the second column and a change of sign. Thus, we proceed directly to the value 5:

	1	−5	4	3	−1
		5	0	20	115
5	1	0	4	23	114

This table shows that 5 is the *least positive* integer upper bound.

We now look for negative zeros/bounds.

	1	−5	4	3	−1
		−1	6	−10	7
−1	1	−6	10	−7	6

The alternation of signs tells us that -1 is a lower bound for real zeros, although it is not a zero itself.

c. There are no rational zeros.

d. There are one or three positive irrational zeros, less than 5, and there is one negative irrational zero, greater than -1.

In example 4–2 F, part 2, we would say that 5 is the *least positive integer upper bound* since no positive integer less than 5 is an upper bound. Similarly -1 is the *greatest negative integer lower bound.*

A historical note

We can solve polynomial functions of degree 0, 1, and 2 (linear and quadratic functions) by way of general formulas and methods. We might ask if there are formulas, like the quadratic formula, for the zeros of polynomial functions of degree greater than 2. Such solutions were sought for thousands of years, and solutions, although much more complicated, were discovered for polynomials of degrees 3 and 4 in the sixteenth century.[4] They were published in the book *Ars magna* by the Italian Geronimo Cardano (1501–1576). Cardano noted that the solution to the general cubic equation was due to Niccolo Tartaglia (ca. 1500–1557).

[4]The methods of exact solution of polynomial functions of degree 3 or 4 are presented in David M. Burton, *The History of Mathematics—An Introduction,* 2 ed. (Dubuque, Iowa: Wm. C. Brown Publishers, 1991).

In 1824, Niels Henrik Abel (1802–1829) of Norway published a proof that there was no similar formula for solving the general polynomial equation of degree 5. Évariste Galois (1811–1832) of France proved that *there is no general method for solving general polynomial equations of degree 5 and above.* This ended the quest for such formulas.

Mastery points

Can you

- List all possible rational zeros of a polynomial function?
- Use synthetic division to find all rational zeros of a polynomial function?
- Use synthetic division to divide a polynomial function by a linear function?
- Use synthetic division and the rational zero theorem to factor certain polynomials?
- Use Descartes' rule of signs to determine the number of possible positive and negative real zeros of a polynomial function?
- Use the bounds theorem to determine least positive upper and greatest negative lower integer bounds for the zeros of a polynomial function?

Exercise 4–2

Find all zeros of the following polynomial functions of degree less than three.

1. $f(x) = 7$ **2.** $g(x) = \frac{1}{2}$ **3.** $f(x) = 12x - 8$ **4.** $g(x) = -5x + 10$
5. $h(x) = x + 11$ **6.** $f(x) = x^2 - 4x - 4$ **7.** $g(x) = -x^2 + 3x + 10$ **8.** $h(x) = x^2 - 8$

List all possible rational zeros for a function defined by the following polynomials.

9. $x^4 - 3x^2 + 6$ **10.** $2x^3 - x - 3$ **11.** $3x^5 - x^3 - 4$ **12.** $x^4 - 8$
13. $6x^2 - 5 + 2x^3$ **14.** $2x^5 - x^6 + 7$ **15.** $3x^4 - 2x^2 + x - 9$ **16.** $5 - 2x^2 + 4x^4$
17. $5x^4 - 2x^3 + 3x - 10$ **18.** $6x^4 - 3x^3 + x - 8$ **19.** $8x^3 - 2x^2 + 4$ **20.** $8x^3 - 3x - 6$
21. $2 - 3x^2 + 4x^3$ **22.** $6 - 2x - 3x^2 + 6x^6$ **23.** $10x^4 - 3x^3 + 2x^2 - 4$ **24.** $4x^4 - 3x^3 + 2x^2 - 10$
25. $8x^3 - 8x + 16$ **26.** $4x^4 - 4x^2 + 4$

Use synthetic division to (a) divide each polynomial by the divisor indicated and (b) to evaluate the function at the value indicated.

27. $f(x) = 3x^4 - 2x^3 + x^2 - 5$
a. $x - 4$ **b.** $f(4)$

28. $g(x) = -2x^3 + 4x^2 - 3x + 1$
a. $x + 2$ **b.** $g(-2)$

29. $f(x) = x^3 - 2x^2 + 3x + 1$
a. $x - 1$ **b.** $f(1)$

30. $h(x) = 3x^4 - x^2 - 6$
a. $x + 1$ **b.** $f(-1)$

31. $h(x) = x^5 - 3x^3 - x^2 + 5$
a. $x + 3$ **b.** $h(-3)$

32. $f(x) = 2x^3 - x - 1$
a. $x - 6$ **b.** $f(6)$

33. $f(x) = \frac{1}{2}x^3 - 3x^2 + \frac{3}{4}x - 3$
a. $x - 6$ **b.** $f(6)$

34. $g(x) = 11x^3 - 2x^2 - 8$
a. $x + 2$ **b.** $g(-2)$

Assume each polynomial below defines a function $f(x)$; for each polynomial

a. Use Descartes' rule of signs to find the number of possible real zeros.
b. List all possible rational zeros of each polynomial.
c. Find all rational zeros; state the multiplicity when greater than one.
d. Write the function as a product of linear and prime quadratic factors if possible.
e. State any irrational zeros found in part d. If there are any other possible irrational zeros state the least positive integer upper bound and the greatest negative integer lower bound for these zeros.

35. $x^4 - x^3 - 7x^2 + x + 6$
36. $x^4 - 5x^2 + 4$
37. $4x^3 - 12x^2 + 11x - 3$
38. $x^4 - 6x^3 + 54x - 81$
39. $x^4 - 8x^3 + 30x^2 - 72x + 81$
40. $x^5 - 4x^3 - 8x^2 + 32$
41. $x^6 - 4x^3 - 5$
42. $6x^3 + 7x^2 - x - 2$
43. $3x^4 + 5x^3 - 11x^2 - 15x + 6$
44. $4x^5 - x^3 - 32x^2 + 8$
45. $x^5 - 4x^2 + 2$
46. $x^3 - 3x^2 - 3$
47. $2x^4 + 6x^3 - 2x - 6$
48. $2x^4 - 5x^3 - 8x^2 + 17x - 6$
49. $3x^4 + 2x^3 - 4x^2 - 2x + 1$
50. $2x^4 - x^3 - 10x^2 + 4x + 8$
51. $x^3 + x^2 - 7x - 10$
52. $2x^3 + x^2 + x - 1$
53. $x^5 - 4x^3 + x^2 + 4x + 4$
54. $x^4 - x^2 + 2x - 2$
55. $9x^4 - 82x^2 + 9$
56. $6x^5 - 23x^4 + 25x^3 - 3x^2 - 7x + 2$
57. $4x^5 + 16x^4 + 37x^3 + 43x^2 + 22x + 4$
58. $x^7 + x^6 - 7x^4 - 7x^3 - 8x - 8$

59. The value $\sqrt[4]{3}$ is a zero of the expression $x^4 - 3$. Use synthetic division to divide $x^4 - 3$ by $x - \sqrt[4]{3}$.

60. The value $\sqrt[5]{6}$ is a zero of the expression $x^5 - 6$. Use synthetic division to divide $x^5 - 6$ by $x - \sqrt[5]{6}$.

61. Use the rational zero theorem to find all possible "rational" zeros of the polynomial $acx^3 + (ad - bc - ace)x^2 + (-ade - bd + bce)x + bde$. Assume that a, b, c, d, and e are integers with no common factors.

62. Use synthetic division and the results of problem 61 to find all the rational zeros of the polynomial $acx^3 + (ad - bc - ace)x^2 + (-ade - bd + bce)x + bde$; use the zeros to factor this polynomial.

63. Referring to the definition in the text of a polynomial of one variable of degree n, define a polynomial function of degree 4. That is, apply the definition when n is 4.

Skill and review

In problems 1–4 graph each function; label all intercepts.

1. $f(x) = (x - 2)^2 - 1$
2. $f(x) = x^2 + x - 4$
3. $f(x) = |x - 2| - 3$
4. $f(x) = x^3 - 1$
5. Find all zeros of $f(x) = 2x^5 + 7x^4 + 2x^3 - 11x^2 - 4x + 4$.
6. Solve $|2x - 3| < 9$.

4–3 The graphs of polynomial functions, and finding zeros of functions by graphical methods

In section 4–2 we studied algebraic methods for finding zeros of polynomial functions. In this section we study the graphs of these functions. We will see that zeros of functions have a clear graphical interpretation, and that graphs provide a powerful tool for finding zeros.

Two very useful pieces of information when graphing polynomial functions are the intercepts of the polynomial and its behavior for large values of x.

Intercepts

A *zero of a function is also an x-intercept of its graph.* This is because x-intercepts are found by replacing y by 0 in an equation. Replacing y by 0 in $y = f(x)$ we solve $0 = f(x)$, which gives the zeros of the function f. Similarly, *the y-intercept of a function is the value of the function when x is zero,* or, for a function f, the value of $f(0)$. This is summarized as follows.

> To find the x-intercepts of a function f, solve $f(x) = 0$.
> To find the y-intercept of a function f, compute $f(0)$.

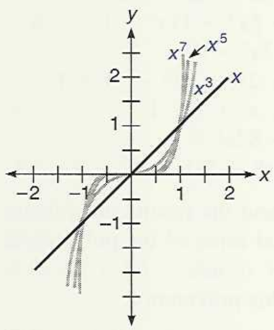


Figure 4–4

Behavior for large values of $|x|$

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5, 6, 7, \ldots$ fall into two categories, shown in figure 4–4 for odd powers and figure 4–5 for even powers. When n is odd, the graph is negative for negative values of x. When n is even the graph is always positive.

The graphs are similar in each figure. Considering polynomial functions with positive coefficient, it is an important fact that for large values of $|x|$ all odd degree polynomial functions behave like the functions in figure 4–4 and all even degree polynomials behave like those in figure 4–5. This is shown in figure 4–6.

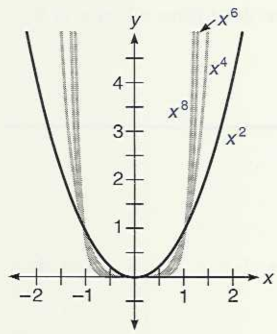


Figure 4–5

To get a feeling for why all polynomials of a certain degree behave in a similar way for large values of $|x|$, consider the values in the following table for the functions $f(x) = x^3$, $f(x) = 2x^3 + x^2$, and $f(x) = x^3 + 5x^2 + 100$.

x	x^3	$x^3 + x^2$	$x^3 + 5x^2 + 100$
−1,000	−1,000,000,000	−999,000,000	−994,999,900
−100	−1,000,000	−990,000	−949,900
−10	−1,000	−900	−400
10	1,000	1,100	1,600
100	1,000,000	1,010,000	1,050,100
1,000	1,000,000,000	1,001,000,000	1,005,000,100

Examination shows that as $|x|$ grows the difference in the three functions becomes smaller and smaller, as a percentage of the value of x^3. A graph of these three functions would become practically indistinguishable as $|x|$ gets larger and larger. Thus, for $|x|$ large enough we might just as well work with the simplest of the functions, $f(x) = x^3$.

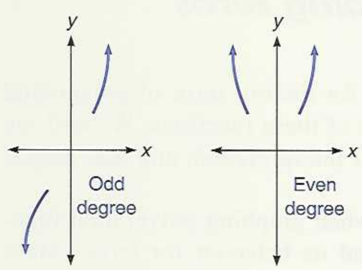


Figure 4–6

Figure 4–6 shows the behavior of odd and even degree polynomials (with positive leading coefficient) for large values of $|x|$; this means somewhat to the right and left of the origin. Near the origin, the behavior varies depending on the values of the zeros of the function.

Graphing using intercepts and behavior for large values of $|x|$

To graph polynomial functions using algebraic properties as a guide, we use several pieces of information, including

- Intercepts.
- The fact that for large values of $|x|$ (i.e., to the right and left of any zeros) all polynomial functions of the same degree are similar (figure 4–6).
- Reflection of the graph about the x-axis when the coefficient of the term of highest degree is negative.
- Plotting points other than the intercepts.

This is illustrated in example 4–3 A.

Of course graphing calculators provide a tool that can achieve similar results, often with much less work. The algebraic properties we are studying still provide an understanding of why a graph behaves as it does, and can be valuable in deciding how to set the RANGE, where to zoom, etc. These functions are graphed on a graphing calculator as shown in earlier sections.

■ Example 4–3 A

Graph the function. Compute all intercepts.

1. $f(x) = (x-1)(x+2)(x-3)$

Since $f(x) = (x-1)(x+2)(x-3) = x^3 - 2x^2 - 5x + 6$, f is a function of degree 3.

x-intercepts:

$0 = (x-1)(x+2)(x-3)$ Let $f(x) = 0$

$x - 1 = 0$ or $x + 2 = 0$ or $x - 3 = 0$ Zero factor property

$x = 1$ or $x = -2$ or $x = 3$ x-intercepts

y-intercept:

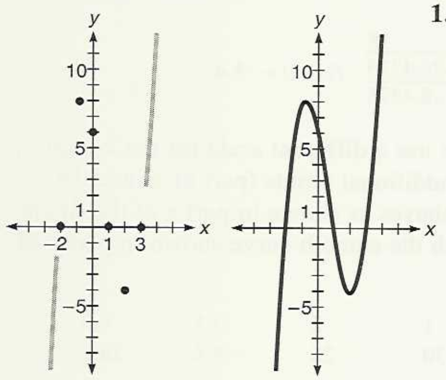
$f(0) = (-1)(2)(-3) = 6$ $y = f(0)$ is the y-intercept

Additional points:

$f(-1) = (-1-1)(-1+2)(-1-3) = 8$; plot $(-1,8)$

$f(2) = (2-1)(2+2)(2-3) = -4$; plot $(2,-4)$

We show the intercepts, the points $(-1,8)$ and $(2,-4)$, and also the similarity with the curve of $y = x^3$ for x to the right and left of the intercepts in part a of the figure. Connecting these with a smooth curve produces the graph of the function in part b.



(a) (b)

Y= (X|T − 1)
(X|T + 2) (
X|T − 3)
RANGE −3,4,−7,10

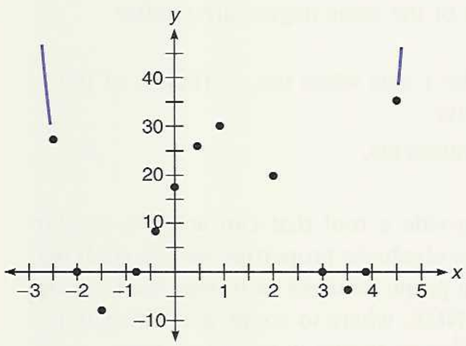
2. $f(x) = x^4 - 4x^3 - 6x^2 + 21x + 18$

y-intercept:

$f(0) = 0^4 - 4(0^3) - 6(0^2) + 21(0) + 18 = 18$ The y-intercept is $f(0)$

x-intercepts: To find x-intercepts we must look for possible rational zeros of f. Use the rational zero theorem to establish that $f(x) = (x + 2)(x - 3)(x^2 - 3x - 3)$.

The zeros of the quadratic $(x^2 - 3x - 3)$ are found by the quadratic formula to be $\dfrac{3 \pm \sqrt{21}}{2}$. Thus, the x-intercepts are -2, 3, and $\dfrac{3 \pm \sqrt{21}}{2}$ (approximately 3.8 and -0.8).



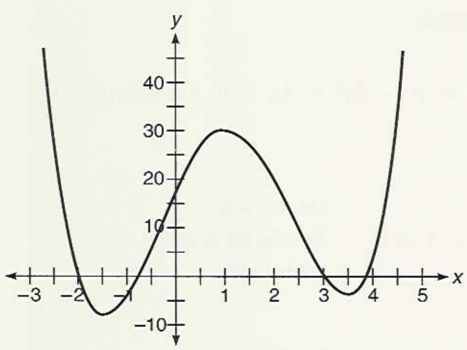
(a)

Additional points: In addition to the x-intercepts and y-intercept we usually need to *choose some additional points to plot* to obtain a graph with approximately the correct proportions. (Unless using a graphing calculator, of course!) As a minimum *choose at least one point between any two intercepts* (-1.5, 1, and 3.5 in the figure), and *one point to the right and one to the left of all intercepts* (-2.5 and 4.5). *Choose additional points when the distance between intercepts is large,* as between -0.8 and 3 in this case.

Synthetic division is a convenient way to compute the function value for these additional points. This is shown for the value -1.5.

	1	-4	-6	21	18
		-1.5	8.25	-3.375	-26.4375
-1.5	1	-5.5	2.25	17.625	-8.4375

$f(-1.5) \approx -8.4$



(b)

With the large range of y-values we use a different scale for the x- and y-axes. We plot the intercepts and additional points (part a). Since the degree of the function is even, it behaves as shown in part a of the figure. This information allows us to sketch the smooth curve shown in part b of the figure.

x	-2.5	-1.5	0.5	1	2	3.5	4.5
y	29.5	-8.4	26.6	30	20	-3.4	36.6

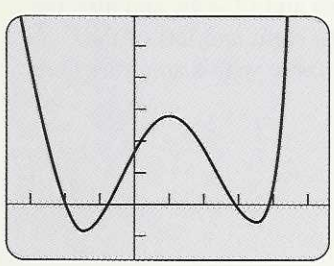
Solution using a graphing calculator

For the TI-81 we would enter

$\boxed{\text{Y=}}$ $\boxed{\text{X|T}}$ $\boxed{\wedge}$ 4 $\boxed{-}$ 4 $\boxed{\text{X|T}}$ $\boxed{\text{MATH}}$ 3 $\boxed{-}$ 6 $\boxed{\text{X|T}}$

$\boxed{x^2}$ $\boxed{+}$ 21 $\boxed{\text{X|T}}$ $\boxed{+}$ 18

$\boxed{\text{RANGE } -3,5,-10,50}$ Yscl=10



The result would look like that shown in the figure. Approximate values of the zeros can be found by tracing and zooming but Newton's method is better.

The TI-81 and Newton's method

There are numeric methods for finding zeros of polynomial functions quickly and to great accuracy by using a programmable calculator and writing a program that searches for a zero of a function. This is useful when the function is well behaved around the zero. For our purposes here we mean that one continuous, smooth line could be used to draw the graph of the function near the zero in question. A good method is called Newton's method. The Texas Instruments TI-81 calculator handbook presents a program called NEWTON that implements this method. Enter the program into the calculator as follows

$\boxed{\text{PRGM}}$ ▶ 1 Program EDIT Mode

Assumes entering the program as Prgm1. Enter the keys that correspond to N E W T O N. For example, T is over the $\boxed{4}$ key.

$\boxed{\text{ENTER}}$

Now type in the program as shown. Use $\boxed{\text{ENTER}}$ after each line.

Program	**Keystroke hints**
:(Xmax − Xmin)/100→D	Xmax is in VARS RNG. Xmin is in VARS RNG. →D is $\boxed{\text{STO▶}}$ $\boxed{x^{-1}}$.
:Lbl 1	Lbl is in PRGM CTL.
:X–Y_1/NDeriv(Y_1,D)→R	Y_1 is in Y-VARS. NDeriv is $\boxed{\text{MATH}}$ 8. ",D" is $\boxed{\text{ALPHA}}$ $\boxed{.}$ $\boxed{\text{ALPHA}}$ $\boxed{x^{-1}}$. →R is $\boxed{\text{STO▶}}$ $\boxed{\times}$.
:If abs (X–R)≤abs (X/1E10)	If is in $\boxed{\text{PRGM}}$ CTL. ≤ is in TEST ($\boxed{\text{2nd}}$ $\boxed{\text{MATH}}$). 1E10 is 1 $\boxed{\text{EE}}$ 10.
:Goto 2	Goto is in $\boxed{\text{PRGM}}$ CTL.
:R→X	Use $\boxed{\text{ALPHA}}$ $\boxed{\times}$ $\boxed{\text{STO▶}}$ $\boxed{\text{X}\mid\text{T}}$.
:Goto 1	
:Lbl 2	
:Disp "ROOT"	Use $\boxed{\text{PRGM}}$ I/O 1 $\boxed{\text{A-LOCK}}$ $\boxed{+}$ R O O T $\boxed{+}$.
:Disp R	Use $\boxed{\text{2nd}}$ $\boxed{\text{CLEAR}}$ when finished entering the program.

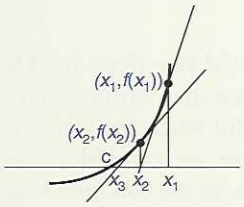


Figure 4–7

Figure 4–7 illustrates how this method gets closer and closer to a root. Assume a function f has a zero at c in the figure. Suppose x_1 is a value of x near c. The program uses the line that is tangent to (i.e., just touches) the function f at the point $(x_1, f(x_1))$ to locate the point x_2, which is closer to c. The program then uses the line that is tangent to the function f at the point $(x_2, f(x_2))$ to locate the point x_3, which is even closer to c. The program repeats this until the difference between the last x-value and the newest x-value is less than a predetermined error value.

The algebraic way in which the program discovers the tangent line at each step is left for a course in calculus. With a little background in this subject, it is not hard to understand.

■ ***Example 4–3 B***

Use the program NEWTON to find the zero near -1 and the zero near 4 in part 2 of example 4–3 A.

Graph the function on the calculator as shown in example 4–3 A.

Use $\boxed{\text{TRACE}}$ to move the cursor near the zero located near -1.

Then select $\boxed{\text{PRGM}}$ 1 (assumes the program NEWTON is stored as the first program). The display shows Prgm1 in the display. Use $\boxed{\text{ENTER}}$ to run the program.

The approximate value of this zero is displayed: $-.7912878475$.

Rerun the program by selecting $\boxed{\text{GRAPH}}$ and then $\boxed{\text{TRACE}}$ again, placing the cursor near the zero near 4. Then select $\boxed{\text{PRGM}}$ 1 $\boxed{\text{ENTER}}$. This will show that this zero is approximately 3.791287847. ■

Graphs at zeros of multiplicity greater than one

The multiplicity of a zero affects the graph. When a zero has **even multiplicity** the function does not cross the x-axis at the corresponding intercept but rather just touches the axis. The reason is discussed after example 4–3 C. A function does cross the x-axis at intercepts corresponding to zeros of **odd multiplicity.**

■ ***Example 4–3 C***

Graph $f(x) = -x^3 + 4x^2 - 5x + 2$.

It would be easiest to think of this as $f(x) = (-1)(x^3 - 4x^2 + 5x - 2)$, and first graph $y = x^3 - 4x^2 + 5x - 2$, then flip the graph about the x-axis.

$y = x^3 - 4x^2 + 5x - 2$

$y = (x - 1)^2(x - 2)$ — Use synthetic division to factor

x-intercepts:

$0 = (x - 1)^2(x - 2)$ — Let $y = 0$ in $y = (x - 1)^2(x - 2)$

$x = 1$ or 2 — 1 has even multiplicity, so the graph just touches but does not cross the x-axis at $x = 1$

y-intercept:

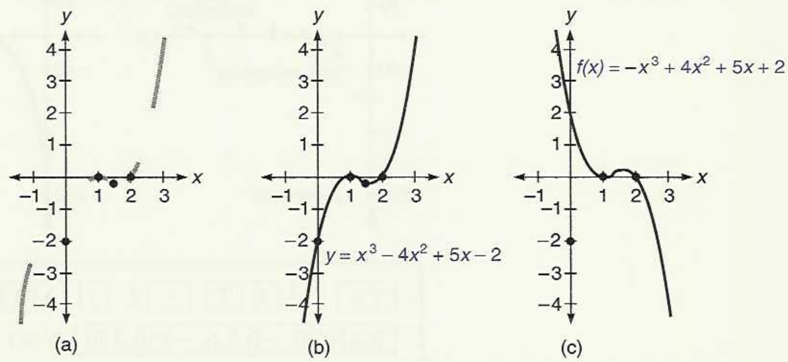
$$y = -2 \qquad \text{Let } x = 0 \text{ in } y = x^3 - 4x^2 + 5x - 2$$

Additional points:

$$y = (1.5 - 1)^2(1.5 - 2) = -0.125 \qquad \text{Let } x = 1.5 \text{ in } y = (x - 1)^2(x - 2)$$

We first plot the intercepts and the additional point $(1.5, -0.125)$. We also know the function behaves like that shown in figure 4–4 for odd-degree functions. We also know that the function crosses the x-axis at the intercept at 2 but not at 1 (part a of the figure).

We next draw a smooth curve to represent $y = x^3 - 4x^2 + 5x - 2$ (part b of the figure), and finally, to obtain the finished graph we draw a graph symmetric to this one about the x-axis (part c of the figure).



$\boxed{\text{Y=}}$ $\boxed{(-)}$ $\boxed{\text{X|T}}$ $\boxed{\text{MATH}}$ 3 $\boxed{+}$ 4 $\boxed{\text{X|T}}$ $\boxed{x^2}$ $\boxed{-}$ 5 $\boxed{\text{X|T}}$ $\boxed{+}$ 2 $\boxed{\text{RANGE } -1,3,-3,4}$

To see why the function in example 4–3 C does not cross the x-axis at 1, consider the equation $y = (x - 1)^2(x - 2)$. When x is less than 1, $x - 1$ is negative. When x is greater than 1, $x - 1$ is positive. However, in both cases $(x - 1)^2$ is positive. Thus, $(x - 1)^2$ provides a constant influence on the sign of the product $(x - 1)^2(x - 2)$ regardless of the value of x. This would be true whatever the exponent of $(x - 1)$, as long as the exponent is even.

Example 4–3 D further illustrates using the multiplicity of a zero to help graph the function, or when using a graphing calculator, to help understand the behavior of the function.

■ Example 4–3 D

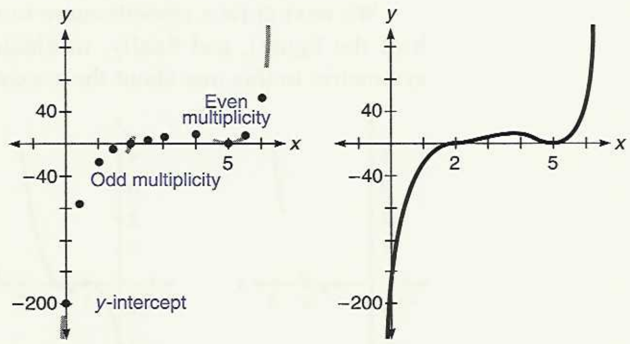
Graph the functions.

1. $f(x) = (x - 2)^3(x - 5)^2$

The only zeros are at 2 (odd multiplicity) and 5 (even multiplicity); the y-intercept is at $f(0) = -200$. Thus the graph does not cross the x-axis at 5, but does cross it at 2. We plot some more points to obtain the graph.

Additional points:

x	0.5	1	1.5	2.5	3	4	5.5	6
y	−68.3	−16	−1.5	0.78	4	8	10.7	64



$\boxed{\text{Y=}}$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{-}$ 2 $\boxed{)}$ $\boxed{\text{MATH}}$ 3 $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{-}$ 5 $\boxed{)}$ $\boxed{x^2}$
$\boxed{\text{RANGE } -0.5, 6, -250, 250}$ YSCL=40

2. $f(x) = (x + 3)^2(x^2 + x + 1)$

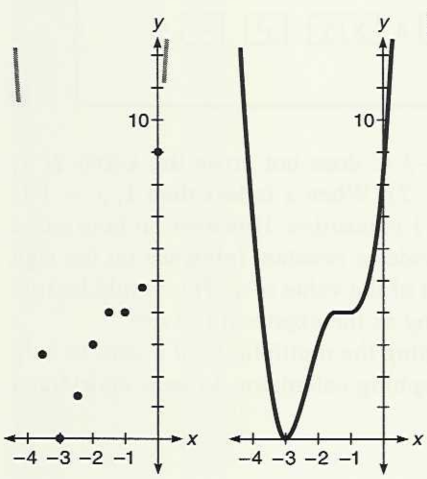
It can be found, by the quadratic formula, that $x^2 + x + 1$ has only complex zeros. Thus, f has only one x-intercept, at -3. This zero is of even multiplicity so the function just touches the axis at -3.

The function f is a fourth-degree polynomial, so for large values of $|x|$ it behaves as indicated in figure 4–5.

The y-intercept is $f(0) = 9$.

Additional points:

x	−4	−3.5	−2.5	−2	−1.5	−1	−0.5	0.5
y	13	2.4	1.2	3	3.9	4	4.7	21.4



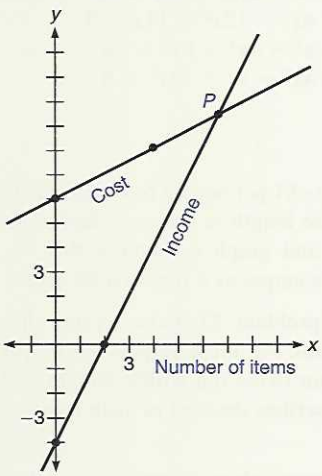
$\boxed{\text{Y=}}$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{+}$ 3 $\boxed{)}$ $\boxed{x^2}$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{x^2}$ $\boxed{+}$ $\boxed{\text{X|T}}$ $\boxed{+}$ 1 $\boxed{)}$
$\boxed{\text{RANGE } -4, 0.5, -0.5, 12}$

Note The shape of this graph in the area of $x = -1$ can be hard to detect without plotting sufficient points. Observe that the complex roots of $x^2 + x + 1$ are $-\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$. It is a good idea to plot additional points at and near the real part of these values, $-\frac{1}{2}$. ■

Applications

If a function is modeling an applied situation in science, business, and the like, then the graph of that function is a powerful tool for answering many types of questions about the situation being modeled.

■ *Example 4–3 E*



A company has discovered that, with respect to a certain product, its income I for i items is $I(i) = 2i - 4$. Its costs for i items is $C(i) = \frac{1}{2}i + 6$.
(a) Graph both of these functions in the same coordinate system, and
(b) determine the minimum number of items that must be produced to make a profit P.

Since it is easier to think in terms of x and y let us graph the relations

$$y = 2x - 4 \quad \text{Income}$$
$$y = \tfrac{1}{2}x + 6, \quad \text{Cost}$$

where x represents the number of items produced.

These are linear functions, so we plot two points for each function, normally the intercepts. For the cost function the x-intercept (-12) is inconvenient, so we plot another point—in this case $x = 4$ so $y = \frac{1}{2}(4) + 6 = 8$.

To make a profit the *income must be greater than cost.* This happens when the number of items, x is to the right of the point P shown in the figure. We find P, the point of intersection, as in section 3–2.

$$y = 2x - 4$$
$$y = \tfrac{1}{2}x + 6$$
$$\text{so } 2x - 4 = \tfrac{1}{2}x + 6$$

Solving for x:

$$2x - 4 = \tfrac{1}{2}x + 6$$
$$4x - 8 = x + 12 \quad \text{Multiply each member by 2}$$
$$3x = 20$$
$$x = 6\tfrac{2}{3}$$

Thus income exceeds cost when the number of items produced is above $6\frac{2}{3}$.

Of course the graphing calculator can also be used to graph both functions. Then tracing along either function will find the approximate (x,y) coordinate where the lines cross. Zooming can be used to increase the accuracy. ■

Mastery points

Can you

- Graph polynomial functions of degree greater than 2 when that polynomial is factored into a product of linear and quadratic factors?
- Construct polynomial functions of any degree that describe stated conditions, and graph these functions?

Exercise 4–3

Graph the following polynomial functions. Label all intercepts.

1. $f(x) = (x - 2)(x + 1)(x + 3)$
2. $g(x) = 2(x - 4)(x^2 - 4)$
3. $h(x) = (x^2 - 1)(x^2 - 9)$
4. $f(x) = (2x - 1)(x - 1)(x - 2)$
5. $g(x) = (x^2 - 4)(4x^2 - 25)(x + 3)$
6. $h(x) = (x^2 - x - 6)(x^2 - 9)(2x + 1)$
7. $f(x) = (x - 1)^2(x + 1)$
8. $g(x) = (x - 2)^2(x + 3)^2$
9. $h(x) = (x + 2)^3(2x - 3)^2$
10. $f(x) = (2x^2 - 3x - 5)^2$
11. $g(x) = (x - 2)^2(x^2 + 3x + 6)$
12. $h(x) = (x - 2)(x^2 + 2x + 5)$
13. $f(x) = x^4 - x^3 - 7x^2 + x + 6$
14. $h(x) = x^4 - 5x^2 + 4$
15. $g(x) = 4x^3 - 12x^2 + 11x - 3$
16. $f(x) = x^4 - 6x^3 + 54x - 81$
17. $f(x) = x^4 - 8x^3 + 30x^2 - 72x + 81$
18. $h(x) = x^5 - 4x^3 - 8x^2 + 32$
19. $g(x) = 6x^3 + 7x^2 - x - 2$
20. $f(x) = 3x^4 + 5x^3 - 11x^2 - 15x + 6$
21. $h(x) = 4x^5 - x^3 - 32x^2 + 8$
22. $g(x) = x^5 - 4x^3 + 8x^2 - 32$

Solve the following problems.

23. A company's income I from a product is described by $I(x) = 3x - 6$, where x is the number of items sold. Its cost for x items is $C(x) = x + 4$.
    a. Graph both of these functions on the same coordinate system.
    b. Determine the minimum number of items that the company must produce to make a profit.

24. A company's income I from a product is described by $I(x) = 4x - 2$, where x is the number of items sold. Its cost for x items is $C(x) = 2x + 4$.
    a. Graph both of these functions on the same coordinate system.
    b. Determine the minimum number of items that the company must produce to make a profit.

25. The Ajax Car Rental Company rents cars for \$30 per day and \$0.30 per mile. The Zeus Car Rental Company rents cars for \$42 per day with unlimited mileage. Under these conditions, the costs of renting from each company for a day can be described as $A(x) = 0.30x + 30$ and $Z(x) = 42$.
    a. Graph both functions in the same coordinate system.
    b. Determine from the graph under which conditions it is cheaper to rent from each company.

26. A photographer is trying to decide whether it is cheaper to use slide film or print film to shoot prints. Slide film costs \$6 for 36 exposures, including developing, and then it costs \$0.25 to make a print from a slide. Print film costs \$10 for 36 exposures, including developing into prints. Under these conditions the cost for x prints, up to 36, for slides s and for prints p is $s(x) = 0.25x + 6$, and $p(x) = 10$.
    a. Graph both functions s and p in the same coordinate system.
    b. Determine the maximum number of good prints that must be kept out of every 36 to make shooting slides cheaper.

27. A rectangular carpet costs \$1 per square foot. It is made in varying widths, but the length is always 3 feet more than the width. Create and graph a function that describes the cost of such a carpet as a function of width.

28. The same company of problem 27 makes carpet that costs \$1.50 per square foot. For these carpets the length must be 1 foot more than twice the width. Create and graph a function that describes the cost of such a carpet as a function of width.

29. A company makes a decorative box of varying sizes, but the proportions of length, width and height are $x + 6$, $x - 2$, and x, respectively, where x is the height in inches. The material that covers the box costs one cent per square inch. Create a function that describes the cost of covering a box as a function of its height. Graph this function.

30. The company described in problem 29 fills these boxes with a different material. Create a function that describes the volume of a box as a function of its height. (The volume of a rectangular solid is the product of its length, width, and height.) Graph this function.

31. Let us call a function "k-scalable" if for any integer k, $k(f(x)) = f(kx)$ for all x in its domain. For example, a function is 2-scalable if $2f(x) = f(2x)$ for all x in its domain.
    a. Show that the function $f(x) = 5x$ is 2-scalable.
    b. Show that the function $f(x) = -3x$ is k-scalable.
    c. Show that the function $f(x) = x^2 - 2x - 8$ is not 3-scalable.

32. Let us call a function "additive" if for all a and b in its domain, $f(a + b) = f(a) + f(b)$.
    a. Show that the function $f(x) = 5x$ is additive.
    b. Show that the function $f(x) = -3x + 1$ is not additive.
    c. Show that the function $f(x) = x^2 - 2x - 8$ is not additive.

Use graphical methods and the programmable capabilities of graphing calculators to find the zeros of the following functions to five decimal places.

33. $f(x) = x^3 - 2x^2 + 5x - 2$

34. $f(x) = 2x^3 - 2x^2 + 5x - 3$

35. $f(x) = 2x^4 - 2x^2 + 5x - 3$

36. $f(x) = x^5 - 3x^3 + 5x^2 - 3$

37. $f(x) = x^5 + 2x^4 + 5x^2 - 3$

38. $f(x) = x^7 - 5x^5 - 2x^4 + 2x^3 + 5x^2 - 1$

39. Napthalene ($C_{10}H_8$) is a very stable chemical. To determine its stability, one needs to solve the equation $x^{10} - 11x^8 + 41x^6 - 65x^4 + 43x^2 - 9 = 0$. It has ten solutions. Find them to five decimal places of accuracy.

40. The equation $x^{10} - 11x^8 + 41x^6 - 61x^4 + 31x^2 - 3 = 0$ must also be solved to fully solve the problem suggested in the previous problem. This equation also has ten solutions. Find them to five decimal places of accuracy.

Skill and review

1. Graph $f(x) = x^2 + 2x - 1$.
2. Solve $\left|\frac{2x-3}{4}\right| \geq \frac{1}{2}$.
3. Solve $x^{-2} - x^{-1} - 12 = 0$.
4. Solve $\sqrt{2x - 2} = x - 5$.
5. Combine $\frac{3}{x-1} - \frac{2}{x+1} + \frac{1}{x}$.
6. Simplify $\sqrt[3]{\frac{4x^2y^7}{3z^8}}$.
7. Rewrite $|5 - 2\pi|$ without absolute value symbols.

4–4 Rational functions

Assume that gasoline costs \$1.00 per gallon, that a car is driven 15,000 miles per year, and that the car's fuel economy is x miles per gallon (mpg). Let y represent the annual savings in fuel costs for increasing the mileage by 10 mpg. Then $y = \frac{15{,}000}{x} - \frac{15{,}000}{x+10}$. Graph this function.

In this section we investigate some of the properties of functions defined as a quotient of two polynomials. These functions are called rational functions. They may be used to study the rate at which a machine can do a job, or to investigate improving gas economy in an automobile, or to describe resistance or capacitance in an electronics circuit.

We begin by defining this class of functions.

Rational function

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomial expressions in the variable x. Unless otherwise stated, the domain of f is all real numbers for which $Q(x) \neq 0$.

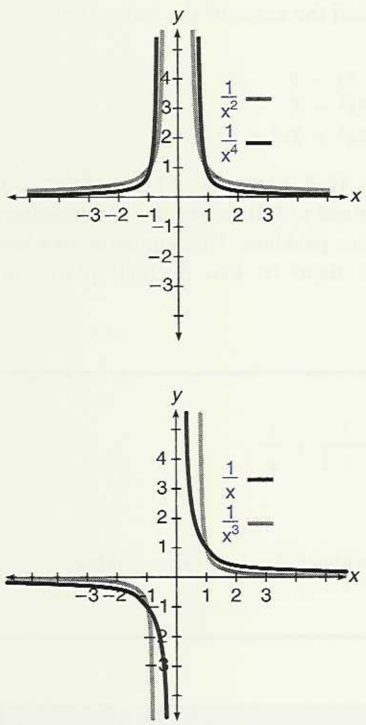


Figure 4–8

Rational functions that are translations of $f(x) = \frac{a}{x^n}$

As with polynomial functions (sections 4–2 and 4–3) we are interested in the graphs of rational functions. A basic rational function is $f(x) = \frac{1}{x}$. We graphed this function in section 3–4; its graph is shown in figure 4–8, which also shows the graph of $f(x) = \frac{1}{x^n}$ for $n = 2, 3, 4$. The graph for $n = 2$ is typical of all even powers, and the graph for $n = 1$ is typical for all odd powers. For the functions in figure 4–8, the y-axis is a **vertical asymptote** and the x-axis is a **horizontal asymptote.** A vertical asymptote is a vertical line that is not part of the graph of the function but that indicates that the function gets larger and larger (or smaller and smaller for negative values) as x gets closer and closer to some given value. A horizontal asymptote is a horizontal line, also not part of the graph of the function, to which the graph of the function gets closer as x gets larger and larger, or smaller and smaller (more and more negative).

We use the following strategy for using algebraic information for graphing rational functions whose graphs are translated and vertically scaled versions of $y = \frac{1}{x^n}$. Of course, graphing calculators can be used; this is illustrated for the TI-81.

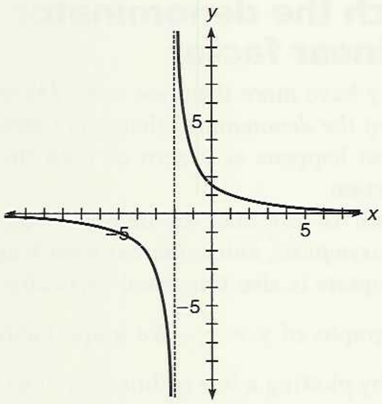
> To graph functions of the form $f(x) = \frac{a}{(x-h)^n} + k$
>
> - There is a vertical asymptote wherever the denominator is zero: at h.
> - The line $y = k$ is a horizontal asymptote.
> - We plot intercepts and a few additional points to determine the basic shape of the graph.

The value a is a vertical scaling factor (section 3–4). It is the primary reason we must plot a few points to obtain an accurate rendition of the graph. The value h represents a horizontal translation, and that of k a vertical translation.

Remember that the y-intercept is at $f(0)$ and the x-intercepts are the solutions to the equation $f(x) = 0$.

■ ***Example 4–4 A***

Describe the function in terms of the graph of $f(x) = \frac{1}{x^n}$ for the appropriate value of n. State all intercepts and asymptotes. Then sketch the graph.



Y= 3 ÷ (X|T
+ 2)
RANGE −10,6,−8,8

1. $f(x) = \dfrac{3}{x+2}$

$y = \dfrac{3}{x-(-2)}$ Rewrite

This graph is the same as that of $y = \dfrac{1}{x}$ except that it is horizontally translated 2 units to the left and vertically scaled by 3.

y-intercept:

$$f(0) = \frac{3}{0+2} = 1\frac{1}{2} : \left(0, 1\frac{1}{2}\right)$$

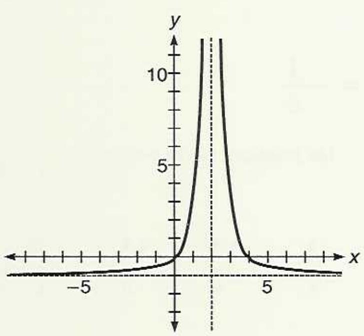
x-intercept:

$$0 = \frac{3}{x+2} \quad \text{Solve } f(x) = 0$$

$$0 = 3 \quad \text{No } x\text{-intercept}$$

Vertical asymptote at $x = -2$ (where the denominator is 0).
Horizontal asymptote at $y = 0$ (the x-axis).
Additional points:

x	-5	-4	-3	-2.5	-1.5	-1	1
y	-1	$-1\frac{1}{2}$	-3	-6	6	3	1



Y= 4 ÷ (X|T
− 2) x^2 − 1
RANGE −5,9,−2,10

2. $g(x) = \dfrac{4}{(x-2)^2} - 1$

This is the graph of $y = \dfrac{1}{x^2}$ translated right 2 units, down 1 unit, and vertically scaled by 4.
Vertical asymptote: $x = 2$
Horizontal asymptote: $y = -1$
y-intercept:

$$g(0) = \frac{4}{(0-2)^2} - 1 = 0$$

x-intercepts:

$$0 = \frac{4}{(x-2)^2} - 1 \quad f(x) = 0$$

$$1 = \frac{4}{(x-2)^2} \quad \text{Add 1 to each member}$$

$$(x-2)^2 = 4 \quad \text{Multiply each member by } (x-2)^2$$

$$x - 2 = \pm 2 \quad \text{Extract the square root of each member}$$

$$x = 2 \pm 2 \quad \text{Add 2 to both members}$$

$$x = 0 \text{ or } 4$$

Additional points:

x	-2	-1	1	1.5	2.5	3	5
y	$-\frac{3}{4}$	$-\frac{5}{9}$	3	15	15	3	$-\frac{5}{9}$

■

Rational functions in which the denominator contains more than one linear factor

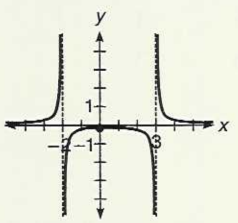
The denominator of a rational function may have more than one zero. *There will be a vertical asymptote for each zero of the denominator* that is not also a zero of the numerator. We will see what happens at a zero of both the numerator and denominator later in this section.

As long as the degree of the numerator is less than the degree of the denominator the *x-axis will be a horizontal asymptote,* unless there is a vertical translation. In that case the horizontal asymptote is also translated vertically.

These graphs cannot be compared to graphs of $y = \frac{a}{x^n}$. We graph them by noting where they have asymptotes and by plotting a few additional points.

■ Example 4–4 B

Graph the function. State all asymptotes and intercepts.



[Y=] 1 [÷] [(] [X|T] [x^2] [−] [X|T] [−] 6 [)]
[RANGE −4,5,−3,3]

1. $f(x) = \dfrac{1}{x^2 - x - 6}$

Factoring the denominator gives $f(x) = \dfrac{1}{(x-3)(x+2)}$.

Vertical asymptotes: $x = 3$ and $x = -2$

Horizontal asymptote: $y = 0$

y-intercept: $\quad f(0) = \dfrac{1}{0^2 - 0 - 6} = -\dfrac{1}{6}$

x-intercepts: $\quad 0 = \dfrac{1}{x^2 - x - 6}$ No solution, so no x-intercepts

Additional points:

x	-3	-2.5	-1.5	1	2	2.5	3.5	4
y	$\frac{1}{6}$	$\frac{4}{11}$	$-\frac{4}{9}$	$-\frac{1}{6}$	$-\frac{1}{4}$	$-\frac{4}{9}$	$\frac{4}{11}$	$\frac{1}{6}$

2. $f(x) = \dfrac{x}{x^2 - 9}$

First, observe that the degree of the numerator (1) is less than the degree of the denominator (2). Thus the x-axis is a horizontal asymptote.

$$f(x) = \frac{x}{(x-3)(x+3)}$$

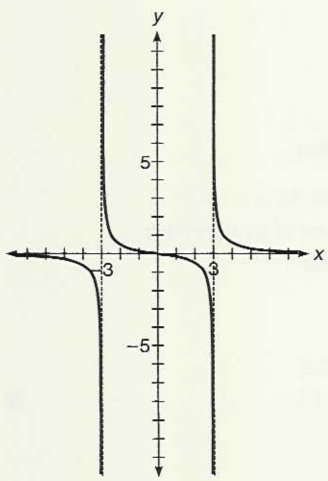
Vertical asymptotes at ± 3

y-intercept:

$$f(0) = \frac{0}{0^2 - 9} = 0$$

x-intercepts:

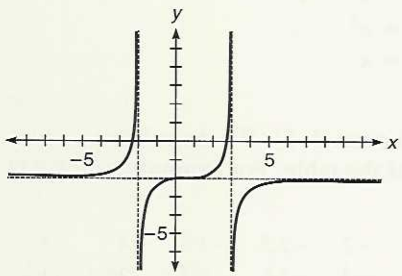
$$0 = \frac{x}{x^2 - 9}$$
$$0 = x$$



[Y=] [X|T] [÷] [(] [X|T] [x^2] [−] 9 [)]
[RANGE −8,8,−5,5]

Additional points:

x	-5	-4	-2	-1	1	2	4
y	$-\frac{5}{16}$	$-\frac{4}{7}$	$\frac{2}{5}$	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{2}{5}$	$\frac{4}{7}$



Y= ((−) X|T + 1) ÷ ((X|T − 3) (X|T + 2)) − 2 RANGE −7,9,−6,3

3. $f(x) = \dfrac{-x + 1}{(x - 3)(x + 2)} - 2$

The degree of the numerator is less than the degree of the denominator, so we expect a horizontal asymptote. Since there is a vertical translation of -2 the horizontal asymptote will be at $y = -2$ instead of the x-axis. There are vertical asymptotes at -2 and 3.

y-intercept: $\quad f(0) = \dfrac{-0 + 1}{(0 - 3)(0 + 2)} - 2 = -2\frac{1}{6}$

x-intercepts:

$$0 = \frac{-x + 1}{(x - 3)(x + 2)} - 2$$

$$2 = \frac{-x + 1}{(x - 3)(x + 2)}$$

$$2(x - 3)(x + 2) = -x + 1$$

$$2x^2 - x - 13 = 0$$

$$x = \frac{1 \pm \sqrt{105}}{4} \approx -2.3, 2.8$$

Additional points:

x	-5	-4	-3	-1	1	2	4	5
y	$-1\frac{3}{4}$	$-1\frac{9}{14}$	$-1\frac{1}{3}$	$-2\frac{1}{2}$	-2	$-1\frac{3}{4}$	$-2\frac{1}{2}$	$-2\frac{2}{7}$

Rational functions in which the degree of the numerator is equal to the degree of the denominator

When the degree of the numerator is *equal to* the degree of the denominator, we first divide the denominator into the numerator, using long division (section 1–2). As illustrated below, this produces a horizontal asymptote.

■ ***Example 4–4 C***

Graph the function $f(x) = \dfrac{x^2}{x^2 - 9}$. State all asymptotes and intercepts.

Observe that the degree of the numerator is equal to the degree of the denominator; therefore divide, using long division (see section 1–2):

$$\begin{array}{r} 1 \\ x^2 - 9 \overline{) \; x^2 + 0x + 0} \\ \underline{-(x^2 \qquad - 9)} \\ 9 \end{array}$$

Long division

$$f(x) = 1 + \frac{9}{x^2 - 9} \text{ or } \frac{9}{(x - 3)(x + 3)} + 1$$

Vertical asymptotes at ± 3; horizontal asymptote at $x = 1$

y-intercept:

$$f(0) = \frac{0^2}{0^2 - 9} = 0$$

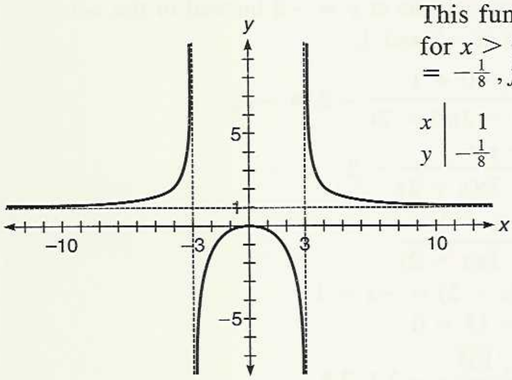
x-intercepts:

$$0 = \frac{x^2}{x^2 - 9}$$
$$0 = x^2$$
$$0 = x$$

Additional points:

This function has y-axis symmetry (see section 3–5). We calculate y values for $x > 0$ and use these to fill in the rest of the table. For example, since $f(1) = -\frac{1}{8}$, $f(-1)$ is also $-\frac{1}{8}$.

x	1	2	2.5	3.5	4	5	−1	−2	−2.5	−3.5	−4	−5
y	$-\frac{1}{8}$	$-\frac{4}{5}$	$-2\frac{3}{11}$	$3\frac{10}{13}$	$2\frac{2}{7}$	$1\frac{9}{16}$	$-\frac{1}{8}$	$-\frac{4}{5}$	$-2\frac{3}{11}$	$3\frac{10}{13}$	$2\frac{2}{7}$	$1\frac{9}{16}$



$\boxed{\text{Y=}}$ $\boxed{\text{X|T}}$ $\boxed{x^2}$ $\boxed{\div}$ $\boxed{(}$
$\boxed{\text{X|T}}$ $\boxed{x^2}$ $\boxed{-}$ 9 $\boxed{)}$
$\boxed{\text{RANGE } -10, 10, -5, 7}$

Rational functions in which the degree of the numerator is greater than the degree of the denominator

When the degree of the numerator is *greater than* the degree of the denominator, we also first divide. In this case, we do not get a horizontal asymptote. If the difference in the degrees is 1 we get a **slant asymptote.**

If the difference in degrees is greater than 1 we get a function of degree 2 or above which the given function approaches when the absolute value of x is large. These are called nonlinear asymptotes. We will restrict ourselves to cases where we get a linear, slant asymptote.

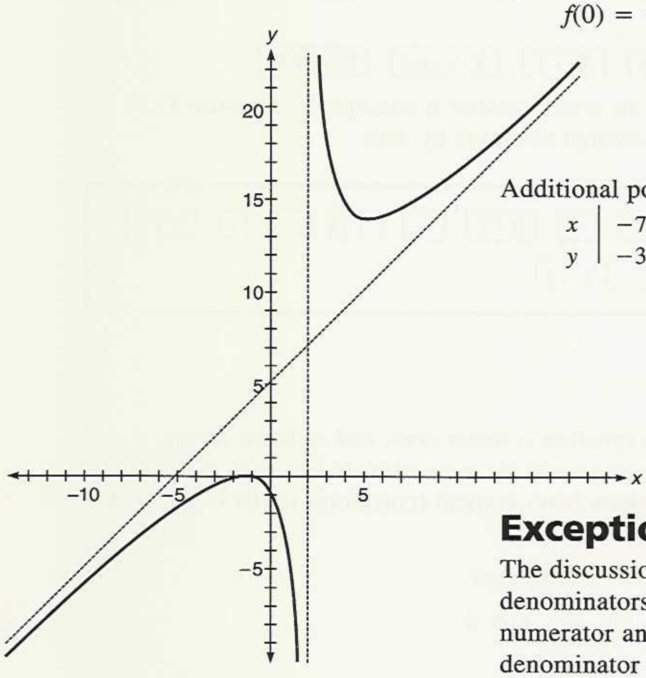
■ ***Example 4–4 D***

Graph the rational function. State all asymptotes and intercepts.

$$f(x) = \frac{x^2 + 3x + 2}{x - 2}$$

$$f(x) = \frac{12}{x - 2} + x + 5 \quad \text{Use long division}$$

The line $y = x + 5$ is a slant asymptote. There is a vertical asymptote at $x = 2$.



y-intercept:

$$f(0) = -1$$

x-intercepts:

$$0 = \frac{x^2 + 3x + 2}{x - 2}$$
$$0 = x^2 + 3x + 2$$
$$0 = (x + 2)(x + 1)$$
$$x = -2 \text{ or } -1$$

Additional points:

x	-7	-5	-3	-1.5	1	1.5	2.5	3	5	7
y	-3.3	-1.7	-0.4	0.07	-6	-17.5	31.5	20	14	14.4

$\boxed{\text{Y=}}$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{x^2}$ $\boxed{+}$ 3 $\boxed{\text{X|T}}$
$+$ 2 $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{-}$ 2 $\boxed{)}$
$\boxed{\text{RANGE } -12,12,-10,25}$

Exceptional cases

The discussion so far has focused on those rational functions with zeros in the denominators. Two other situations are worth noting. The first is where the numerator and denominator have a common factor. The second is where the denominator is never zero. These two cases are illustrated in example 4–4 E.

■ ***Example 4–4 E***

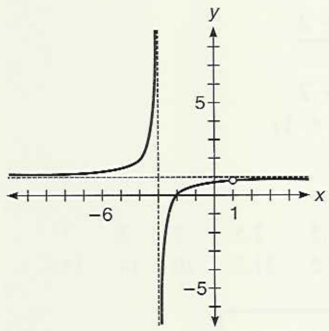
Graph the function.

1. $f(x) = \dfrac{x^2 + x - 2}{x^2 + 2x - 3}$

Factoring gives $f(x) = \dfrac{(x - 1)(x + 2)}{(x - 1)(x + 3)}$, which is not defined at $x = 1$ and $x = -3$, and otherwise reduces to $\dfrac{x + 2}{x + 3}$. Thus, the function f can be described as $f(x) = \dfrac{x + 2}{x + 3}$, $x \neq 1$.

This function falls into the category of the degree of the numerator equal to the degree of the denominator, so we graph it as illustrated in example 4–4 C.

To show that the function is not defined at 1, we show a hole in the graph for $x = 1$. When this function is graphed on a graphing calculator the hole at $x = 1$ will not necessarily be visible. This is because of the

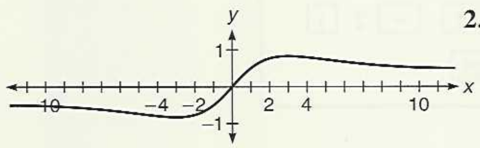


limitations of technology. However, it can cause problems. To illustrate this enter the function into Y_1 and try to evaluate $f(x)$ at $x = 1$. Do this as follows:

1 [STO▶] [X | T] [Y-VARS] [ENTER]

This sequence produces an error, because it attempts to compute Y_1 at $x = 1$, which causes an attempt to divide by zero.

[Y=] [(] [X|T] [x^2] [+] [X|T] [−] 2 [)] [÷] [(] [X|T] [x^2] [+] 2 [X|T] [−] 3 [)]



[Y=] [(] 5 [X|T] [−] 1 [)] [÷] [(] [X|T] [x^2] [+] 9 [)]
RANGE −12,12,−2,2

2. $f(x) = \dfrac{5x - 1}{x^2 + 9}$

The denominator of this function is never zero, and so there are no vertical asymptotes. *The degree of the numerator is less than the degree of the denominator* and there is no vertical translation *so the x-axis is a horizontal asymptote.*

y-intercept: $y = -\frac{1}{9}$ x-intercept: $x = \frac{1}{5}$

Additional points:

x	−10	−5	−4	−2	−1	1	2	4	5	10
y	−0.47	−0.76	−0.84	−0.85	−0.6	0.4	0.69	0.76	0.7	0.45

Mastery points

Can you

- Graph a rational function of the form $\dfrac{a}{(x-h)^n} + k$, and compare its graph to the graph of $y = \dfrac{1}{x^n}$ for a suitable value of n?
- Graph a general rational function when the degree of the numerator is less than the degree of the denominator by analyzing its vertical and horizontal asymptotes and point plotting?
- Graph a rational function when the degree of the numerator is greater than or equal to the degree of the denominator by first doing long division?

Exercise 4–4

Describe the function in terms of the graph of $f(x) = \frac{1}{x^n}$ for an appropriate value of n. Then graph the function. State all intercepts and asymptotes.

1. $f(x) = \frac{3}{x-2}$

2. $g(x) = \frac{1}{x+3}$

3. $h(x) = \frac{-2}{x+4}$

4. $f(x) = \frac{2}{x-1}$

5. $f(x) = \frac{3}{(x-2)^2}$

6. $f(x) = \frac{2}{(x+3)^3}$

7. $g(x) = \frac{1}{(x+\frac{1}{2})^4}$

8. $g(x) = \frac{4}{(x-1)^2}$

9. $h(x) = \frac{3}{(x-2)^3}$

10. $h(x) = \frac{-4}{(x+5)^4}$

11. $f(x) = \frac{-4}{(x-2)^2}$

12. $g(x) = \frac{-2}{x^3}$

13. $h(x) = \frac{1}{x-1} + 2$

14. $g(x) = \frac{1}{(x+1)^2} - 2$

15. $f(x) = \frac{1}{(x-3)^2} + 2$

16. $g(x) = \frac{1}{x+3} + 3$

Graph the following rational functions. State all intercepts and asymptotes.

17. $f(x) = \frac{3}{x^2 - 3x - 18}$

18. $h(x) = \frac{1}{x^2 - x - 20}$

19. $g(x) = \frac{-4}{(x-2)(x+4)}$

20. $f(x) = \frac{-1}{x^2 - 1}$

21. $h(x) = \frac{2x-1}{x^2 - 4}$

22. $g(x) = \frac{x}{x^2 - 9}$

23. $h(x) = \frac{2x}{x^2 - 4x - 5}$

24. $g(x) = \frac{-3x+1}{x^2 - x - 12}$

25. $f(x) = \frac{-2x-3}{x^2 - 4x}$

26. $f(x) = \frac{x}{x^2 - 6x + 5}$

27. $g(x) = \frac{3x^2 - 1}{(x-2)(x^2-9)}$

28. $h(x) = \frac{x-3}{(x-1)(x^2 - 2x - 8)}$

29. $h(x) = \frac{3x-1}{(x^2 - 2x)(x+1)}$

30. $g(x) = \frac{2x+3}{x^2 - 16}$

31. $h(x) = \frac{2}{(x-1)(x+3)} + 2$

32. $f(x) = \frac{x}{(x+2)(x-1)} - 2$

33. $g(x) = \frac{x}{x^2 - 2x - 15} - 4$

34. $h(x) = \frac{1}{x^2 + 5x + 4} + 1$

Graph the following rational functions. State all intercepts and asymptotes.

35. $f(x) = \frac{x}{x+1}$

36. $g(x) = \frac{2x}{x-2}$

37. $h(x) = \frac{-2x}{x-3}$

38. $f(x) = \frac{-x}{x-1}$

39. $f(x) = \frac{x^2 - 3}{x^2 - 4x - 5}$

40. $g(x) = \frac{x^2 - x + 1}{x^2 + 3x - 4}$

41. $h(x) = \frac{-3x^2 + 2x - 1}{x^2 - 4}$

42. $f(x) = \frac{4x^2 - 1}{x^2 - 1}$

Graph the following rational functions. State all intercepts and asymptotes.

43. $f(x) = \frac{\frac{1}{2}x^2}{x-1}$

44. $g(x) = \frac{x^3}{x^2 - 1}$

45. $h(x) = \frac{x^3}{x^2 - 2x + 1}$

46. $f(x) = \frac{x^2}{x+2}$

47. $h(x) = \frac{x^3 - x}{x^2 - 4}$

48. $h(x) = \frac{3x^3 - x^2 - 2}{x^2 - x - 6}$

49. $g(x) = \frac{x^3 + 8}{x^2 + 3x - 4}$

50. $f(x) = \frac{3x^4 - x^3}{(x-1)(x^2-4)}$

51. $g(x) = \frac{2x^4 - x^2}{(x+2)(x^2-1)}$

52. $f(x) = \frac{-x^3 - x^2 + 11x - 9}{x^2 + 2x - 8}$

Graph the following rational functions. State all intercepts and asymptotes.

53. $f(x) = \dfrac{x^2 - 1}{x^2 + 2x - 3}$

54. $f(x) = \dfrac{x^2 - x - 6}{x - 3}$

55. $g(x) = \dfrac{x - 2}{x^3 - x^2 - 4x + 4}$

56. $h(x) = \dfrac{x^3 - 2x^2 - 25x + 50}{(x - 5)(x^2 - 7x + 10)}$

$\boxed{57.}$ $h(x) = \dfrac{x^3 + 4x^2 + 3x + 12}{x^2 + 3}$

58. $f(x) = \dfrac{x + 1}{2x^2 + x - 1}$

59. $f(x) = \dfrac{8x - 2}{x^2 + 1}$

60. $g(x) = \dfrac{2x + 1}{x^4 + 1}$

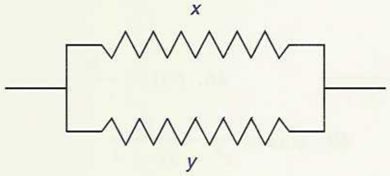
61. $g(x) = \dfrac{x^2 - 4}{x^2 + 4}$

62. $f(x) = \dfrac{x^3 - 1}{x^2 + 1}$

63. Iron ore is being moved 120 feet by a conveyor belt that travels at r feet per minute, and then an additional 160 feet by another belt that is 5 feet per minute slower than the first. Since distance equals the product of rate and time, or $d = rt$, then $t = \dfrac{d}{r}$. Thus the total time T taken to move the ore, as a function of the rate r of the first belt, is $T(r) = \dfrac{120}{r} + \dfrac{160}{r - 5}$, or $T(r) = \dfrac{280r - 600}{r(r - 5)} = 40\left(\dfrac{7r - 15}{r(r - 5)}\right)$. Graph the relation $y = \dfrac{7x - 15}{x(x - 5)}$.

64. It takes a certain machine m minutes to print a certain number of pages. Because of a fixed warm-up time, a second machine always takes $m + 2$ minutes to print the same number of pages. Under these conditions the rate at which these machines print pages when running together is $\dfrac{1}{m} + \dfrac{1}{m + 2} = \dfrac{2m + 2}{m(m + 2)}$. Graph the relation $y = \dfrac{2x + 2}{x(x + 2)}$.

65. In an electronics circuit with two resistances, x and y in parallel (see the figure), and where the total resistance is to be 20 ohms, the relation $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{20}$ will be true.
(a) Solve this relation for y as a function of x (note that $x > 0$ and $y > 0$ are reasonable assumptions to make).
(b) Graph this function.



66. If x represents the value, in ohms, of a certain resistor in an electronics circuit, and a second resistor in parallel with it is 10 ohms more, then the reciprocal of their combined resistance y is given by $y = \dfrac{1}{x} + \dfrac{1}{x + 10}$, or $y = \dfrac{2x + 10}{x^2 + 10x}$. Graph this relation.

67. In problem 66 the relation described the reciprocal of the combined resistance. The actual resistance would be described by the relation $y = \dfrac{x^2 + 10x}{2x + 10}$. Graph this relation.

68. Assume that gasoline costs \$1.00 per gallon, that a car is driven 15,000 miles per year, and that the car's fuel economy is x miles per gallon (mpg). Let y represent the annual savings in fuel costs for increasing the mileage by 10 mpg. Then $y = \dfrac{15{,}000}{x} - \dfrac{15{,}000}{x + 10}$. Combine the right member into one term, then graph this function. Use a scale of 5,000 per unit on the y-axis.[5]

69. Use the formula of problem 68 to compute the annual savings obtained by increasing the mileage of a car by 10 mpg if the car now gets (a) 10 mpg, (b) 20 mpg, and (c) 30 mpg.

[5]This problem presented by Floyd Vest, North Texas State University, in *Consortium*, by COMAP, Arlington, Mass., Summer 1990.

Skill and review

If $f(x) = 2x - 3$ and $g(x) = x^2 + 2x + 3$, compute:

1. $f(5)$ **2.** $g(-4)$ **3.** $f(g(2))$ **4.** $g(f(-1))$

5. Solve $x = 2y + 7$ for y.

6. Solve $x = \dfrac{1}{y - 2}$ for y.

7. Graph $f(x) = 2x^4 - x^3 - 14x^2 + 19x - 6$.

8. Solve $|4 - 3x| = 16$.

4–5 Composition and inverse of functions

A company charges \$0.50 per cubic foot for a plastic it makes. Thus, if x is the number of cubic feet of this plastic, the price paid in dollars is $P(x) = \frac{1}{2}x$. The volume of a cube is $V(x) = x^3$, where x is the length of one of its dimensions. Let C be the cost function that will give the cost of a cube of this plastic when the length of a side is x feet. Compute an expression for $C(x)$.

The process used to compute the expression for C in the preceding problem is called composing functions. It is one of the operations that can be performed on functions that we study in this section.

The basic operations for functions

The concept of a function is so important in advanced mathematics that an algebra for functions has developed. This algebra is a system for performing computations in which the elements are functions, not numbers. The operations of addition/subtraction and multiplication/division are easy to define for functions.

Definition of addition, subtraction, multiplication, and division of functions

Let f and g be functions. Then, for every element x in the domain of both functions

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f\cdot g)(x) = f(x)\cdot g(x)$$
$$(f/g)(x) = \frac{f(x)}{g(x)},\ \text{if } g(x) \neq 0$$

Note that "$f+g$," "$f-g$," etc. are the names of new functions.

■ ***Example 4–5 A***

Find expressions for $f+g$, $f-g$, $f\cdot g$, and f/g for the given functions.

1. $f(x) = x^2 - 3$, $g(x) = x$

$$(f+g)(x) = f(x) + g(x) \quad \text{Definition of } (f+g)(x) \text{ is } f(x)+g(x)$$
$$= (x^2-3) + x \quad \text{Replace } f(x) \text{ by } x^2-3, \text{ and } g(x) \text{ by } x$$
$$= x^2 + x - 3 \quad \text{Simplify}$$

$$(f-g)(x) = f(x) - g(x) = (x^2-3) - x = x^2 - x - 3$$
$$(f\cdot g)(x) = f(x)\cdot g(x) = (x^2-3)(x) = x^3 - 3x$$
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{x^2-3}{x} \text{ if } x \neq 0$$

2. $f(x) = \sqrt{x-3}$, $g(x) = \sqrt{x+1}$

$$(f+g)(x) = f(x) + g(x) = \sqrt{x-3} + \sqrt{x+1}$$
$$(f-g)(x) = f(x) - g(x) = \sqrt{x-3} - \sqrt{x+1}$$
$$(f \cdot g)(x) = f(x) \cdot g(x) = \sqrt{x-3} \cdot \sqrt{x+1} = \sqrt{x^2 - 2x - 3}$$
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x-3}}{\sqrt{x+1}}$$

Composition of functions

There is an operation defined for functions, called **composition** of functions, which does not have a direct analogy with arithmetic, as do the operations defined above. It gives us a new way to look at functions.

Consider this function: $f(x) = \sqrt{x^2 - x - 3}$. How would one compute $f(4)$? First, we would evaluate $x^2 - x - 3$ for $x = 4$. This is 9. We would then compute $\sqrt{9}$, giving 3. Thus, $f(4) = 3$. We have viewed it as a two-stage operation. We can formalize this idea as composition of functions.

Composition of two functions *f* and *g*

Let f and g be functions, $x \in$ domain of g, $g(x) \in$ domain of f. Then

$$(f \circ g)(x) = f(g(x))$$

Note $(f \circ g)$ is read "f composed with g."

First we focus on how this operation is used. It is important to look again at $f(x)$ notation. Consider as an example $f(x) = 2x^2 - x + 1$. Whatever replaces x in $f(x)$ also replaces x in $2x^2 - 3x + 1$. Consider the following sequence of examples using this function.

$f(x)$	$= 2x^2$	$- 3x$	$+ 1$
$f(1)$	$= 2(1)^2$	$- 3(1)$	$+ 1$
$f(a)$	$= 2a^2$	$- 3a$	$+ 1$
$f(c+2)$	$= 2(c+2)^2$	$- 3(c+2)$	$+ 1$
$f(x^4-2)$	$= 2(x^4-2)^2$	$- 3(x^4-2)$	$+ 1$
$f(g(x))$	$= 2(g(x))^2$	$- 3(g(x))$	$+ 1$

The statement $f(x) = 2x^2 - x + 1$ can be viewed as a pattern; any meaningful mathematical expression can replace the x in the entire statement.

■ ***Example 4–5 B***

Compute an expression for $(f \circ g)(x)$ and $(g \circ f)(x)$ for the given functions.

1. $f(x) = x^2 - 3x + 2$; $g(x) = 2x - 5$

$(f \circ g)(x) = f(g(x))$		Definition of $(f \circ g)(x)$ is $f(g(x))$
	$= [g(x)]^2 - 3[g(x)] + 2$	Replace x by $g(x)$ in $x^2 - 3x + 2$
	$= [2x-5]^2 - 3[2x-5] + 2$	Replace $g(x)$ by $2x - 5$
	$= 4x^2 - 26x + 42$	Simplify

Thus, $(f \circ g)(x) = 4x^2 - 26x + 42$.

$(g \circ f)(x) = g(f(x))$	Definition of $(g \circ f)(x)$ is $g(f(x))$
$= 2(f(x)) - 5$	Replace x by $f(x)$ in $2x - 5$
$= 2(x^2 - 3x + 2) - 5$	Replace $f(x)$ by $x^2 - 3x + 2$
$= 2x^2 - 6x - 1$	Simplify

Thus, $(g \circ f)(x) = 2x^2 - 6x - 1$.

2. $f(x) = \sqrt{x^2 - 3}$ and $g(x) = \dfrac{1}{x}$

$(f \circ g)(x) = f(g(x))$	
$= \sqrt{[g(x)]^2 - 3}$	$f(x) = \sqrt{x^2 - 3}$, so $f(g(x)) = \sqrt{[g(x)]^2 - 3}$
$= \sqrt{\left(\dfrac{1}{x}\right)^2 - 3}$	Replace $g(x)$ by $\dfrac{1}{x}$
$= \sqrt{\dfrac{1 - 3x^2}{x^2}}$	$\left(\dfrac{1}{x}\right)^2 - 3 = \dfrac{1}{x^2} - \dfrac{3x^2}{x^2} = \dfrac{1 - 3x^2}{x^2}$
$(g \circ f)(x) = g(f(x))$	
$= \dfrac{1}{f(x)}$	$g(x) = \dfrac{1}{x}$ so $g(f(x)) = \dfrac{1}{f(x)}$
$= \dfrac{1}{\sqrt{x^2 - 3}}$.	Replace $f(x)$ by $\sqrt{x^2 - 3}$ ■

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. This subset of the domain of g can be difficult to find, and we will not pursue this problem in this text except to illustrate the difficulty as follows.

Consider the functions $f(x) = \sqrt{12 - x^2}$, and $g(x) = \sqrt{x - 4}$. Then

$$(f \circ g)(x) = f(g(x)) = \sqrt{12 - (g(x))^2} = \sqrt{12 - (\sqrt{x - 4})^2} = \sqrt{16 - x}$$

The implied domain of $(f \circ g)(x) = \sqrt{16 - x}$ is $x \leq 16$. This includes the value 0, for example. However, since $g(0)$ is not defined, $f(g(0))$ is not defined, so 0 is not in the domain of $f \circ g$. In complicated situations the expression for $f \circ g$ cannot be relied on to determine its domain.

Inverses of functions

Consider $f(x) = 2x + 3$, and $g(x) = \dfrac{x - 3}{2}$ (see figure 4–9). By computation we could determine the following facts.

$$f(1) = 5 \text{ and } g(5) = 1$$
$$f(2) = 7 \text{ and } g(7) = 2$$
$$f(-5) = -7 \text{ and } g(-7) = -5$$

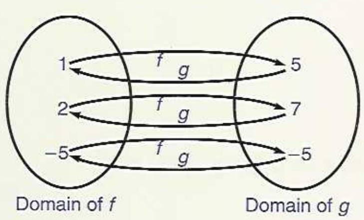


Figure 4–9

Whatever value z we try, f sends z to some value z', and g sends z' back to z (see figure 4–10). In fact we can prove this; let z represent any real number.

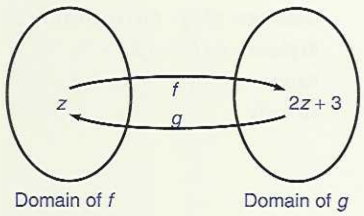


Figure 4–10

Then,

$$f(z) = 2z + 3 \text{ and}$$
$$g(f(z)) = g(2z + 3) = \frac{(2z + 3) - 3}{2} = z$$

Also

$$g(z) = \frac{z - 3}{2} \text{ and}$$
$$f(g(z)) = f\left(\frac{z - 3}{2}\right) = 2\left(\frac{z - 3}{2}\right) + 3 = z$$

When two functions f and g act this way we say they are inverse functions.

> **Function inverse**
> If $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ for all x in the domains of functions f and g, then f and g are said to be inverse functions. The inverse of a function f is symbolized as f^{-1}.

Thus, if $f(x) = 2x + 3$ we can say that $f^{-1}(x) = \frac{x - 3}{2}$. Note that the superscript -1, when applied to the name of a function, is not an exponent; it does not indicate division, as it does if applied as an exponent of a real valued expression. Thus, $f^{-1}(x)$ **does not mean** $\frac{1}{f(x)}$.

> **To show that two functions *f* and *g* are inverses of each other**
> Show that [1] $(f \circ g)(x) = x$ and [2] $(g \circ f)(x) = x$

It is possible for one and not the other of equations [1] and [2] to be true. For example, take $f(x) = x^2$ and $g(x) = \sqrt{x}$. $(f \circ g)(x) = (\sqrt{x})^2 = x$, but $(g \circ f)(x) = \sqrt{x^2} = |x|$, not x.

■ ***Example 4–5 C***

Show that f and g are the inverse functions of each other.

1. $f(x) = \frac{1}{3}x - 1$
$g(x) = 3x + 3$
$(f \circ g)(x) = f(g(x)) = \frac{1}{3}(3x + 3) - 1 = x$
$(g \circ f)(x) = g(f(x)) = 3(\frac{1}{3}x - 1) + 3 = x$

2. $f(x) = \sqrt{x}$
$g(x) = x^2, x \geq 0$
$(f \circ g)(x) = f(g(x)) = \sqrt{x^2} = |x|$, but since $x \geq 0$ for g, $|x| = x$.
$(g \circ f)(x) = g(f(x)) = (\sqrt{x})^2 = x$ ■

The following two theorems are quite useful in working with functions and their inverses.

> **Ordered pairs reverse in f^{-1}**
> If an ordered pair (a,b) is in a function f, and if f has an inverse function f^{-1}, then (b,a) is an ordered pair in f^{-1}.

To see this, note that $f(a) = b$, so that $(a,b) \in f$, and $f^{-1}(b) = f^{-1}(f(a)) = a$, so that $(b,a) \in f^{-1}$.

> **Only one-to-one functions have inverse functions**
> A function f has an inverse function f^{-1}, if and only if it is one to one.

The following explains why only one-to-one functions have inverse functions.

If a function f is not one to one, then there are at least two elements in it in which the ordered pairs have the same second component. Let these points be (x_1,b) and (x_2,b). Now if f^{-1} exists, then (b,x_1) and (b,x_2) are elements of that function. However, these points have a first element that repeats, making f^{-1} a relation, but not a function.

Similarly, if f is a one-to-one function, then no second element repeats. Therefore, the relation created when we reverse all the ordered pairs is a function, since there is no repetition of first elements, and this function meets the definition of the inverse of f. Thus we determine that the above theorem is true.

The graph of a function's inverse

The fact that the ordered pairs reverse in the inverse function of a function means that *the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.* By way of example, observe the graphs of the functions in the last example. These are shown in figure 4–11. To draw a graph that is symmetric about the line $y = x$ to a given graph, we draw lines perpendicular to the line $y = x$, as shown, and plot points at equal distances from this line, but on the other side of this line. Since the ordered pairs of f all reverse in f^{-1} *the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.*

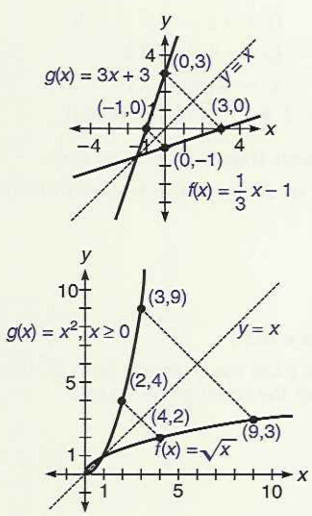


Figure 4–11

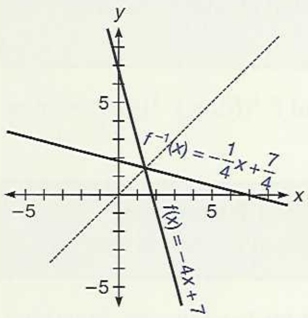
Finding an expression for the inverse of a function

The fact that the ordered pairs reverse in a function's inverse provides a method that can be used to find the inverse of a function.

> **To find $f^{-1}(x)$ for a one-to-one function f**
> - Replace $f(x)$ by y.
> - Replace each x by y and y by x (this is "reversing" the ordered pairs of f).
> - Solve for y.
> - Replace y by $f^{-1}(x)$.

This method is useful when the resulting expression can be solved for y. It is illustrated in example 4–5 D.

■ ***Example 4–5 D***



Find f^{-1} for each function f. Also, graph f and f^{-1}.

1. $f(x) = -4x + 7$

First note that the graph of this function is a straight line that passes the horizontal line test (see section 3–5), and thus is one to one. It thus has an inverse function.

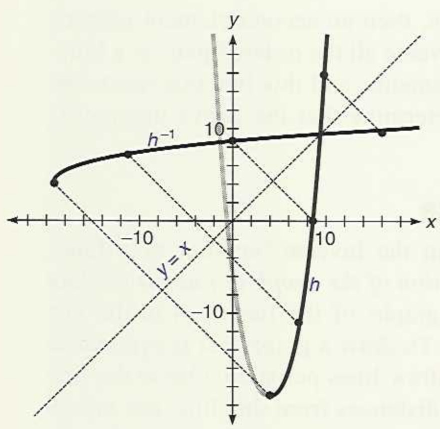
$y = -4x + 7$	Replace $f(x)$ by y (this relation describes f)
$x = -4y + 7$	Replace x by y and y by x (this relation describes f^{-1})
$4y = -x + 7$	Solve for y
$y = -\frac{1}{4}x + \frac{7}{4}$	Divide each member by 4
$f^{-1}(x) = -\frac{1}{4}x + \frac{7}{4}$	Rewrite with $y = f^{-1}(x)$

Thus, $f^{-1}(x) = -\frac{1}{4}x + \frac{7}{4}$.

The graphs of f and f^{-1} are both straight lines that are graphed by plotting their intercepts.

Intercepts for f: $(0,7)$, $(\frac{7}{4},0)$

Intercepts for f^{-1}: $(7,0)$, $(0,\frac{7}{4})$



2. $h(x) = x^2 - 8x - 3,\ x \geq 4$

The graph of h is a parabola. We graph it by completing the square.

$$y = x^2 - 8x + 16 - 16 - 3$$
$$y = (x - 4)^2 - 19$$

Vertex at $(4,-19)$.

y-intercept:

$$h(0) = 0^2 - 8(0) - 3 = -3$$

x-intercepts:

$$0 = (x - 4)^2 - 19$$
$$(x - 4)^2 = 19$$
$$x - 4 = \pm\sqrt{19}$$
$$x = 4 \pm \sqrt{19} \approx -0.4,\ 8.4$$

With $x \geq 4$ we can see by the horizontal line test that h is one to one.

We draw the graph of h^{-1} from the graph of h, by reflecting points in h about the line $y = x$.

We proceed to find an expression for h^{-1}.

$y = x^2 - 8x - 3$ and $x \geq 4$	$y = h(x)$
$x = y^2 - 8y - 3$ and $y \geq 4$	Interchange x and y
$0 = y^2 - 8y + (-x - 3)$ and $y \geq 0$	To solve for y use the quadratic formula; this requires that the equation be set to 0

$$y = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-x - 3)}}{2(1)}$$
$$= \frac{8 \pm \sqrt{64 + 4(x + 3)}}{2}$$
$$= 4 \pm \sqrt{x + 19} \text{ and } y \geq 4$$

$y = 4 + \sqrt{x + 19}$ since $y \geq 4$, and $4 - \sqrt{x + 19}$ is less than or equal to 4

Thus, $h^{-1}(x) = 4 + \sqrt{x + 19}$. ■

The graphing calculator can help verify that we have found the correct expression for the inverse of a function. We graph the original function and its inverse in the same graph. If one is the mirror image of the other across the line $y = x$, then we have correctly found the inverse.

Mastery points

Can you

- Find expressions for $f + g$, $f - g$, $f \cdot g$, and f/g when given expressions that define functions f and g?
- Compute an expression for $(f \circ g)(x)$ and $(g \circ f)(x)$ when given expressions that define functions f and g?
- Demonstrate that two functions are inverses of each other?
- Find f^{-1} when given a function f, and graph both functions?

Exercise 4–5

Find expressions for $f + g$, $f - g$, $f \cdot g$, f/g, and $(f \circ g)(x)$ and $(g \circ f)(x)$ for the given functions f and g.

1. $f(x) = 3x - 5$; $g(x) = -2x + 8$

2. $f(x) = 2x + 3$; $g(x) = \frac{1}{2}x - 3$

3. $f(x) = x + 4$; $g(x) = \sqrt{x - 4}$

4. $f(x) = x^2 - 3$; $g(x) = \sqrt{8 - x}$

5. $f(x) = \dfrac{x - 3}{2x}$; $g(x) = \dfrac{x}{x - 1}$

6. $f(x) = x^3 - 3x^2 + x - 4$; $g(x) = x^2 - 1$

7. $f(x) = x^4 - x^2 + 3$; $g(x) = \sqrt{\dfrac{x}{x + 1}}$

8. $f(x) = -x$; $g(x) = x$

9. $f(x) = x$; $g(x) = 3$

10. $f(x) = 2$; $g(x) = 3$

11. $f(x) = \sqrt[3]{x - 5}$; $g(x) = x^3 + 5$

12. $f(x) = \dfrac{x}{x^2 - 2x - 15}$; $g(x) = \dfrac{1}{x}$

(a) Show that the following functions f and g are inverses of each other. Assume the domains as indicated are correct.
(b) Graph each function and its inverse in the same coordinate system.

13. $f(x) = 2x - 7$; $g(x) = \frac{1}{2}x + 3\frac{1}{2}$

14. $f(x) = -\frac{1}{3}x + \frac{1}{2}$; $g(x) = -3x + \frac{3}{2}$

15. $f(x) = \frac{1}{3}x + \frac{8}{3}$; $g(x) = 3x - 8$

16. $f(x) = x - 1$; $g(x) = x + 1$

17. $f(x) = x^2 - 9, x \geq 0$; $g(x) = \sqrt{x + 9}$

18. $f(x) = \sqrt{4 - 2x}$; $g(x) = 2 - \frac{1}{2}x^2, x \geq 0$

19. $f(x) = x^3$; $g(x) = \sqrt[3]{x}$

20. $f(x) = x^3 - 3$; $g(x) = \sqrt[3]{x + 3}$

21. $f(x) = x^2 - 2x + 3, x \geq 1$; $g(x) = \sqrt{x - 2} + 1$

22. $f(x) = \sqrt{x + 9} - 2$; $g(x) = x^2 + 4x - 5, x \geq -2$

23. $f(x) = \dfrac{2x}{x - 3}$; $g(x) = \dfrac{3x}{x - 2}$

24. $f(x) = \dfrac{x - 3}{x - 2}$; $g(x) = 2 - \dfrac{1}{x - 1}$

Find the inverse function of the given function. All the functions are one to one.

25. $f(x) = 4x - 5$

26. $g(x) = \dfrac{3x - 2}{4}$

27. $h(x) = 12 - \frac{5}{2}x$

28. $f(x) = \dfrac{x}{2}$

29. $g(x) = x^2 - 9, x \geq 0$

30. $h(x) = x^2 + 3, x \geq 0$

31. $f(x) = \sqrt{9 - x^2}, x \geq 0$

32. $g(x) = \sqrt{2x^2 - 16}, x \geq 0$

33. $h(x) = \sqrt{x - 4}$

34. $f(x) = \sqrt{5 - x}$

35. $g(x) = 2x^3 - 9$

36. $h(x) = x^3 + 20$

37. $f(x) = \sqrt[3]{4x - 5}$ **38.** $g(x) = \sqrt[3]{1 - x} + 3$ $\boxed{39.}$ $f(x) = \dfrac{3 - 5x}{4x}$

40. $g(x) = \dfrac{x}{3 - x}$ **41.** $g(x) = \dfrac{x}{x + 1}$ **42.** $h(x) = \dfrac{2x - 3}{x}$

43. $h(x) = \dfrac{x - 1}{x + 1}$ **44.** $f(x) = \dfrac{1 - 2x}{x - 3}$ **45.** $h(x) = x^2 - 2x - 9, x \geq 1$

46. $f(x) = x^2 - 8x + 1, x \geq 4$ $\boxed{47.}$ $g(x) = 2x^2 + 3x - 2, x \geq -\frac{3}{4}$ **48.** $h(x) = 3x^2 - 6x - 1, x \geq 1$

49. A company charges \$0.50 per cubic foot for a plastic it makes. Thus, if x is the number of cubic feet of this plastic the price paid in dollars is $P(x) = \frac{1}{2}x$. The volume of a cube is $V(x) = x^3$, where x is the length of one of its dimensions. Thus, if x is in feet, then the cost function $C = P \circ V$ will give the cost of a cube of this plastic when the length of a side is x feet. Compute an expression for $C(x)$.

50. The volume of a sphere is $V = \frac{4}{3}\pi^3$, where r is the radius of the cube. Using the price function from problem 49 compute an expression for $C(x)$, the cost of a sphere of radius r composed of the same plastic.

51. A railroad car is accelerating slowly so that its forward velocity after t seconds, in feet per second, is $R(t) = \frac{1}{4}t$. A person in the car is walking in the opposite direction so that the person's velocity relative to the car is $V_r(t) = 2$ feet per second. Under these circumstances the person's velocity relative to the earth is $V_e = R - V_r$. Compute an expression for $V_e(t)$.

52. The velocity, relative to the earth, of an aircraft flying with the wind is $V_e = V_a + V_w$, where V_a is the velocity of the aircraft relative to the air and V_w is the velocity of the wind. Find an expression for the velocity of the aircraft after t seconds, $V_e(t)$, when $V_w(t) = 25$ (miles per hour) and $V_a(t) = 80 + \frac{1}{10}t$ (miles per hour).

53. A stunt person runs off a platform horizontally at 5 feet per second. The falling person describes a parabola. A movie director wants the fall to be in front of a curtain. The area of the curtain is important to the director because its cost is a function of its area. Under these conditions the stunt person's horizontal distance traveled after t seconds is $d_h(t) = 5t$ feet, and the vertical distance fallen is $d_v(t) = 16t^2$ feet. The area of the curtain is therefore $A = d_h d_v$. Find an expression for $A(t)$. Note that the units for A is square feet.

54. The cost of the material for the curtain of problem 53 in dollars for x square feet is $C(x) = 0.5x$. Under these circumstances the cost to the director for the curtain for a fall of t seconds is $C \circ A$. (a) Find an expression for $(C \circ A)(t)$ and (b) use this expression to predict the cost of the curtain for a 3-second fall.

55. The area of a rectangle with width 4 and length $x + 4$, $x \geq 0$, is $A(x) = 4(x + 4)$. Find the inverse of A, which would give the value of x for a given area.

56. A falling object with no initial vertical velocity falls a distance $d(t) = 16t^2$ feet in t seconds, $t \geq 0$. Find the inverse of this function, which would give the time necessary to fall a distance d.

57. In an electronic circuit in which two resistances are in parallel, and the value of the resistances are 20 ohms and x ohms, the total resistance is $R(x) = \dfrac{20x}{20 + x}$. Find the inverse of this function, which would give the value of x required for a total resistance R.

58. $C(t) = \frac{5}{9}(t - 32)$ gives the centigrade temperature for a given temperature t in degrees Fahrenheit. Find the inverse of this function, which would find the Fahrenheit temperature for a given temperature in degrees centigrade.

59. Show that $f(x) = x^2 - 9$, $x \leq 0$ and $g(x) = -\sqrt{x + 9}$ are inverse functions. You will need to recall that $\sqrt{x^2} = |x|$, $(\sqrt{x})^2 = x$, and the definition of $|x|$ (section 1–1) to do this problem properly.

60. Show that $f(x) = 3 - \sqrt{x + 14}$ and $g(x) = x^2 - 6x - 5$, $x \leq -3$ are inverse functions. See problem 59.

61. Find the inverse of the general linear function $f(x) = ax + b$, where a and b are real-valued constants. Under what conditions would this function not have an inverse?

$\boxed{62.}$ Find the expression for the inverse of the general quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, if $x \geq \dfrac{-b}{2a}$.

63. It is possible to find the inverse of certain functions by "undoing" the operations implied in their definition. For example, consider $f(x) = 2x - 3$. To calculate a value when given a value for x, we (1) multiply by 2 and then (2) subtract 3.

To undo this we should (1) add 3 and then (2) divide by 2. This would be $f^{-1}(x) = \dfrac{x+3}{2}$, or $f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$.

Another example is $f(x) = \frac{1}{5}x + 7$. To evaluate for a given x we (1) divide by 5 and then (2) add 7. To undo we (1) subtract 7 and then (2) multiply by 5. This would be $f^{-1}(x) = 5(x - 7)$, or $f^{-1}(x) = 5x - 35$.

This method is useful whenever the variable x appears only once in an expression. Use this method to find the inverse function for problems 25 through 38.

Skill and review

1. Combine $\dfrac{2}{x+3} - \dfrac{3}{x-2}$.
2. Graph $f(x) = \dfrac{2}{x+3}$.
3. Graph $f(x) = \dfrac{2}{(x+3)(x-1)}$.
4. Graph $f(x) = \dfrac{2x^2}{(x+3)(x-1)}$.
5. Solve $\left|\dfrac{5x-2}{x+1}\right| < 2$.
6. Graph $f(x) = x^3 - x^2 - x + 1$.

4–6 Decomposition of rational functions

Find the sum $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{99 \cdot 100}$.

A programmable calculator could be programmed to compute this sum, but it turns out that a little algebra will do the same job faster and more accurately. In this section we introduce the algebra necessary for this task.

Calculation will show that $\dfrac{1}{x-1} + \dfrac{3}{x-4}$ can be combined into $\dfrac{4x-7}{(x-1)(x-4)}$. We sometimes need to be able to decompose a rational expression like $\dfrac{4x-7}{(x-1)(x-4)}$ back into a sum of two or more fractions. Besides solving the problem posed above, this finds a great deal of use in advanced mathematics, such as the Calculus, and in discrete mathematics.

We first consider the case where we can factor the denominator into a product of linear factors. We do this by assuming the existence of certain values, as shown in the examples. The rational expression is said to be decomposed into **partial fractions.**

Linear factors in the denominator

Example 4–6 A illustrates the procedure for decomposing a rational expression in which

- the degree of the numerator is less than the degree of the denominator and
- the denominator factors into a product of linear factors.

The procedure is to assume a separate rational expression for each linear factor, with some unknown numerator.

■ ***Example 4–6 A***

Decompose the rational expression into partial fractions: $\dfrac{4x - 7}{x^2 - 5x + 4}$

$$\frac{4x - 7}{(x - 1)(x - 4)}$$ Factor the denominator

Note that this is the expression we saw above, so we know the answer—of course, we will assume we don't.

$$\frac{4x - 7}{(x - 1)(x - 4)} = \frac{A}{x - 1} + \frac{B}{x - 4}$$ Assume a separate rational expression for each factor of the denominator

Now solve for A and B by first multiplying each member by the LCD $(x - 1)(x - 4)$.

$$\frac{(x - 1)(x - 4)}{1} \cdot \frac{4x - 7}{(x - 1)(x - 4)}$$
$$= \frac{(x - 1)(x - 4)}{1} \cdot \frac{A}{x - 1} + \frac{(x - 1)(x - 4)}{1} \cdot \frac{B}{x - 4}$$

[1] $4x - 7 = A(x - 4) + B(x - 1)$

A useful technique for finding A and B is to let $x = 4$ and then $x = 1$ in equation [1]. These are the values for which one of the factors is zero.

[1] $4x - 7 = A(x - 4) + B(x - 1)$

$4(4) - 7 = A(4 - 4) + B(4 - 1)$ Let $x = 4$

$9 = 0A + 3B$

$9 = 3B$

$3 = B$

[1] $4x - 7 = A(x - 4) + B(x - 1)$

$4(1) - 7 = A(1 - 4) + B(1 - 1)$ Let $x = 1$

$-3 = -3A + 0B$

$-3 = -3A$

$1 = A$

Now, $A = 1$ and $B = 3$, so

$$\frac{4x-7}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4}$$
$$= \frac{1}{x-1} + \frac{3}{x-4}$$

It can be shown that *if one of the linear factors of the denominator is of multiplicity greater than one it must be assumed that it appears once for each positive integer up to and including its multiplicity.* This is illustrated in example 4–6 B.

■ ***Example 4–6 B***

Decompose the rational expression into partial fractions: $\dfrac{3x^2 - 15x + 14}{(x-1)(x-2)^2}$

We assume the existence of values A, B, C such that

$$\frac{3x^2 - 15x + 14}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Multiply both members by the LCD $(x-1)(x-2)^2$:

[1] $\quad 3x^2 - 15x + 14 = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$

Now find A, B, and C.

Let $x = 2$ in equation [1]:

$$3(2)^2 - 15(2) + 14 = A(2-2)^2 + B(2-1)(2-2) + C(2-1)$$
$$-4 = C(1)$$
$$-4 = C$$

Let $x = 1$ in equation [1]:

$$3(1)^2 - 15(1) + 14 = A(1-2)^2 + B(1-1)(1-2) + C(1-1)$$
$$2 = A(-1)^2$$
$$2 = A$$

To obtain B we can let x be any value other than 1 and 2 and use the known values for A and C; 0 is a logical value to use for x.

Let $x = 0$ in equation [1]:

$$3(0)^2 - 15(0) + 14 = A(0-2)^2 + B(0-1)(0-2) + C(0-1)$$
$$14 = 4A + 2B - C$$
$$14 = 4(2) + 2B - (-4) \qquad A = 2, C = -4$$
$$1 = B$$

Thus, the solution is $\dfrac{3x^2 - 15x + 14}{(x-1)(x-2)^2} = \dfrac{2}{x-1} + \dfrac{1}{x-2} + \dfrac{-4}{(x-2)^2}$. ■

Another very important point is: *If the degree of the numerator is greater than or equal to that of the denominator we must do long division first.*

■ ***Example 4–6 C***

Decompose $\dfrac{2x^2 - 3x + 7}{(x-1)(x+2)}$ into partial fractions.

$(x-1)(x+2) = x^2 + x - 2$ Multiply the denominator

$$\begin{array}{r} 2 \\ x^2 + x - 2\overline{)2x^2 - 3x + 7} \\ \underline{2x^2 + 2x - 4} \\ -5x + 11 \end{array}$$ Divide using long division

Thus, $\dfrac{2x^2 - 3x + 7}{(x-1)(x+2)} = 2 + \dfrac{-5x + 11}{(x-1)(x+2)}$.

$\dfrac{-5x + 11}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$ Assume the values *A*, *B*

Solving produces $A = 2$, $B = -7$, so $\dfrac{2x^2 - 3x + 7}{(x-1)(x+2)} = 2 + \dfrac{2}{x-1} + \dfrac{-7}{x+2}$. ■

Prime quadratic factors in the denominator

Recall (from section 4–2) that a quadratic expression $ax^2 + bx + c$, $a \neq 0$, is prime (over the real numbers) if the discriminant $b^2 - 4ac < 0$. If the denominator of a rational expression has quadratic factors that are prime, we employ a similar procedure to that shown above.

The difference in procedures is that we must assume the numerators are of the form $ax + b$, and not simply constants as before. Here we restrict ourselves to cases where the prime quadratic factors appears once—this is because the process of finding the values of the assumed variables can become very complicated otherwise.

■ ***Example 4–6 D***

Decompose the rational expression into partial fractions:

$\dfrac{x^2 + 3x + 2}{(x^2 + x + 1)(x - 1)}$

We assume values *A*, *B*, *C* such that $\dfrac{x^2 + 3x + 2}{(x^2 + x + 1)(x - 1)} = \dfrac{Ax + B}{x^2 + x + 1}$ $+ \dfrac{C}{x-1}$ and proceed as before.

Multiply by the LCD $(x^2 + x + 1)(x - 1)$:

[1] $x^2 + 3x + 2 = (Ax + B)(x - 1) + C(x^2 + x + 1)$

Let $x = 1$: $1 + 3 + 2 = (A + B)(0) + C(1 + 1 + 1)$
$2 = C$

We now let x take on two other values to obtain two equations with which to find *A* and *B*:

Let $x = 0$: $2 = B(-1) + 2(1)$ $C = 2$
$0 = B$

Let $x = -1$: $1 - 3 + 2 = (-A)(-2) + 2(1)$ $B = 0, C = 2$
$-1 = A$

The solution is $\dfrac{x^2 + 3x + 2}{(x^2 + x + 1)(x - 1)} = \dfrac{-x}{x^2 + x + 1} + \dfrac{2}{x - 1}$. ■

Mastery points

Can you

- Decompose rational expressions in which the denominator is a product of linear factors into partial fractions?
- Decompose rational expressions in which the denominator is a product of prime quadratic and linear factors into partial fractions (assuming the factors are limited to exponents of one)?

Exercise 4–6

Decompose the expressions into partial fractions.

1. $\dfrac{x - 10}{x^2 - 5x + 4}$ **2.** $\dfrac{7x + 5}{x^2 - 2x - 15}$ **3.** $\dfrac{-6x - 2}{x^2 - 1}$

4. $\dfrac{x^3 - 13x - 30}{x^2 - 9}$ **5.** $\dfrac{4x^3 - 6x^2 - 1}{2x^2 - 3x + 1}$ **6.** $\dfrac{4x^3 - 12x^2 + 7x + 14}{2x^2 - 5x - 3}$

7. $\dfrac{18x^3 - 51x^2 + 14x + 28}{6x^2 - 13x - 5}$ **8.** $\dfrac{11x + 4}{6x^2 - 37x + 6}$ **9.** $\dfrac{3x^2 - 4x - 1}{(x - 1)^2(x - 2)}$

10. $\dfrac{2x^2 - 5x + 1}{(x - 1)^2(x - 2)}$ **11.** $\dfrac{13x^2 - 12x + 12}{2x^2(x - 2)}$ **12.** $\dfrac{4x^2 + 3x - 12}{(x + 4)(x^2 - 16)}$

13. $\dfrac{3x^3 - 11x^2 + x - 17}{(x - 3)^2(x + 1)^2}$ **14.** $\dfrac{7x^2 - 2x + 23}{(x - 3)^2(x + 1)^2}$ **15.** $\dfrac{-5x^3 + 32x^2 - 17x - 22}{(x - 3)^2(x + 1)^2}$

16. $\dfrac{x^3 - 4x^2 + 13x + 2}{(x - 3)^2(x + 1)^2}$ **17.** $\dfrac{x^2 + 4x + 4}{(x - 1)(x^2 + x + 1)}$ **18.** $\dfrac{x^2 + 6x + 12}{(x + 2)(x^2 + 3x + 3)}$

19. $\dfrac{-x^2 + 3x - 4}{x^3 + 2x^2 + 4x}$ **20.** $\dfrac{5x^2 + x + 2}{(x + 1)(x^2 + 1)}$ **21.** $\dfrac{x^2 - 11x - 2}{(x - 3)(x^2 + x + 1)}$

22. $\dfrac{2x^2 - 2x + 8}{(x^2 + 5)(x + 1)}$ **23.** $\dfrac{x^2 + 6x - 5}{(x + 3)(x^2 + 2x + 4)}$ **24.** $\dfrac{31x + 12}{(x - 3)(x^2 + 3x + 3)}$

25. In an electronics circuit with two resistances in parallel, the reciprocal of the combined resistances is $\dfrac{2x + 15}{x^2 + 15x + 50}$. Decompose this term using partial fractions.

26. In a situation in which one machine requires x hours to produce 5 items and a second machine requires $x + 3$ hours to do the same thing, the rate at which both machines work is $\dfrac{2x + 3}{x^2 + 3x}$. Decompose this term using partial fractions.

27. In the sum $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{99 \cdot 100}$, the nth term is of the form $\dfrac{1}{n(n + 1)}$. Decompose this term using the method of partial fractions and use the result to compute the sum.

28. In the sum $\dfrac{2}{1 \cdot 3} + \dfrac{2}{3 \cdot 5} + \dfrac{2}{5 \cdot 7} + \cdots + \dfrac{2}{99 \cdot 101}$, the nth term is of the form $\dfrac{2}{n(n + 2)}$. Decompose this term using the method of partial fractions and use the result to compute the sum.

29. In the sum $\frac{2}{1 \cdot 3} + \frac{2}{2 \cdot 4} + \frac{2}{3 \cdot 5} + \cdots + \frac{2}{99 \cdot 101}$, the nth term is of the form $\frac{2}{n(n+2)}$. Decompose this term using the method of partial fractions and use the result to compute the sum (see problem 28).

30. In the sum $\frac{3}{1 \cdot 4} + \frac{5}{4 \cdot 9} + \frac{7}{9 \cdot 16} + \cdots + \frac{199}{99^2 \cdot 100^2}$, the nth term is of the form $\frac{2n+1}{n^2(n+1)^2}$. Decompose this term using the method of partial fractions and use the result to compute the sum.

Skill and review

1. Compute **a.** 8^3 **b.** $8^{1/3}$ **c.** 8^{-3} **d.** $8^{-1/3}$.
2. If $2^5 = a^5$, what is a?
3. If $2^a = 2^5$, what is a?
4. Graph $f(x) = 2x^2 - x - 6$.
5. Graph $f(x) = (x - 1)(x + 2)(x - 2)$.
6. Solve $x^3 - x^2 + 1 > x$.
7. Graph $f(x) = \frac{x^2 + 1}{x^2 - 1}$.

Chapter 4 summary

- **Quadratic function** A function of the form $f(x) = ax^2 + bx + c$, $a \neq 0$; its graph is a parabola.
- **Vertex form for a quadratic function** $f(x) = a(x - h)^2 + k$. The vertex is at (h,k); it opens up if $a > 0$, down if $a < 0$.
- **Polynomial function** A function of the form $f(x) = a_nx^n + \cdots + a_2x^2 + a_1x + a_0$, $a_n \neq 0$.
- If $f(c) = 0$ for some function f and a real number c then c is a zero of the function.
- **Rational zero theorem** If $\frac{p}{q}$ is a rational number in lowest terms (p and q are therefore integers) and $\frac{p}{q}$ is a zero of a polynomial function, then p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.
- **Remainder theorem** If f is a nonconstant polynomial function and c is a real number, then the remainder when $x - c$ is divided into $f(x)$ is $f(c)$.
- **Multiplicity of zeros** If $(x - c)^n$, $n \in N$, divides a function f, and $(x - c)^{n+1}$ does not divide f, we say that c is a root of multiplicity n.
- If a zero of a polynomial is of **even multiplicity** the graph just touches, but does not cross, the x-axis at that point. If the zero is of **odd multiplicity** the function crosses the axis at the zero.
- Every polynomial of positive degree n over R is the product of a real number and one or more prime linear or quadratic polynomials over R.
- **Bounds theorem for real zeros** Let c be a real number and $f(x)$ be a polynomial function with real coefficients and positive leading coefficient; consider all the coefficients in the last line of the synthetic division algorithm as applied to the value c. Then c is

  an upper bound if $c \geq 0$ and these coefficients are all positive or zero.

  a lower bound if $c \leq 0$ and these coefficients alternate between nonnegative and nonpositive values.
- **Descartes' rule of signs** Let $f(x)$ be a function defined by a polynomial with real coefficients. Then,

  the number of *positive* real zeros is equal to the number of variations in sign in $f(x)$ or is less than this number by a multiple of 2.

  the number of *negative* real zeros is equal to the number of variations in sign in $f(-x)$ or is less than this number by a multiple of 2.
- **Rational function** A function of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial expressions in the variable x.
- **Vertical asymptote** A vertical line that a function gets closer and closer to as x approaches a certain value.
- **Horizontal asymptote** A horizontal line that a function gets closer and closer to as $|x|$ gets larger and larger.
- **Slant asymptote** A slanted straight line that a function gets closer and closer to as $|x|$ gets larger and larger.
- In general a rational function has a vertical asymptote wherever the denominator takes on the value zero.

- **Graphing rational functions** To graph rational functions we use the following information.

  Horizontal and slant asymptotes

  Vertical asymptotes

  Intercepts

  Plotting points
- **Arithmetic operations for functions** Let f and g be functions and x a value in the domain of f and g. Then

  $(f + g)(x) = f(x) + g(x)$

  $(f - g)(x) = f(x) - g(x)$

  $(f \cdot g)(x) = f(x) \cdot g(x)$

  $(f/g)(x) = \dfrac{f(x)}{g(x)}$, if $g(x) \neq 0$
- **Composition of functions** Let f and g be functions, $x \in$ domain of g, $g(x) \in$ domain of f. Then $(f \circ g)(x) = f(g(x))$.
- **Inverse functions** If $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ for all x in the domains of functions f and g, then f and g are said to be inverse functions.
- A function f has an inverse function f^{-1}, if and only if it is one to one.
- If an ordered pair (a,b) is in a function f, and if f has an inverse function f^{-1}, then (b,a) is an ordered pair in f^{-1}.
- The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.
- **To find $f^{-1}(x)$ for a one-to-one function f:**

  Replace $f(x)$ by y.

  Replace each x by y and y by x.

  Solve for y.

  Replace y by $f^{-1}(x)$.

Chapter 4 review

[4–1] Graph the following parabolas. Compute the intercepts and vertex.

1. $y = x^2 - 3x - 18$ **2.** $y = -3x^2$
3. $y = x^2 - 4x$ **4.** $y = -x^2 + 5x + 6$
5. $y = 9 - x^2$ **6.** $y = 3x^2 + 4x - 4$
7. $y = x^2 - 5x - 1$ **8.** $y = x^2 + 2x + 2$
9. $y = x^2 - x + 5$ **10.** $y = -x^2 - 4x$

Solve problems 11–14 by creating an appropriate second-degree equation and finding the vertex.

11. A homeowner has 200 feet of fencing to fence off a rectangular area behind the home. The home will serve as one boundary (the fence is only necessary for three of the four sides). What are the dimensions of the maximum area that can be fenced off, and what is the area?

12. What is the area of the largest rectangle that can be created with 400 feet of fence? What are the length and width of this rectangle?

13. Suppose that 400 feet of fencing are available to fence in an area. Will a half-circle (perimeter including the straight side) or a rectangle contain a greater area? (*Hint:* For a fixed perimeter the radius of a half-circle is a fixed value, and can be found. See the result of problem 12 also.)

14. If an object is thrown vertically into the air with an initial velocity of v_0 ft/s, then its distance above the ground s, for time t, is given by $s = v_0t - 16t^2$. Suppose an object is thrown upward with initial velocity 512 ft/s; find how high the object will go and when it will return to the ground.

Graph the following functions.

15. $h(x) = \begin{cases} -2x - 1, & x < -1 \\ x + 2, & x \geq -1 \end{cases}$

16. $g(x) = \begin{cases} x + 2, & x < 1 \\ 2x^2 - 4x + 5, & x \geq 1 \end{cases}$

[4–2] List all possible rational zeros for the following polynomials.

17. $2x^4 - 3x^2 + 6$ **18.** $2x^3 - 4x^2 + 2x - 10$
19. $x^5 - x^3 - 4$ **20.** $3x^4 - 8$

Use synthetic division to (a) divide each polynomial by the divisor indicated and (b) to evaluate the function at the value indicated.

21. $f(x) = 2x^4 - 5x^3 + 2x^2 - 1$
a. $x - 3$ **b.** $f(3)$

22. $g(x) = -2x^3 - 3x^2 - 3x + 2$
a. $x + 4$ **b.** $g(-4)$

23. $f(x) = \frac{1}{2}x^3 - 3x^2 + \frac{3}{4}x - 3$
a. $x - 4$ **b.** $f(4)$

For each function:

a. Use Descartes' rule of signs to find the number of possible real zeros.
b. List all possible rational zeros of each polynomial.
c. Find all rational zeros; state the multiplicity when greater than one.
d. Factor each as much as possible over the real numbers.
e. If there are any possible irrational zeros state the least positive integer upper bound and the greatest negative integer lower bound for these zeros.

24. $f(x) = 2x^4 + x^3 - 24x^2 - 9x + 54$
25. $g(x) = 2x^4 - x^3 - 9x^2 + 4x + 4$
26. $h(x) = 2x^4 - 9x^3 - 13x^2 + 4x + 4$
27. $f(x) = 16x^5 - 48x^4 - 40x^3 + 120x^2 + 9x - 27$

[4–3] Graph the following polynomial functions. State all intercepts.

28. $g(x) = \frac{1}{4}(x - 4)(x^2 - 16)$
29. $h(x) = (x^2 - 4)(4x^2 - 9)$
30. $g(x) = (x^2 - 9)(4x^2 - 9)(x - 3)$
31. $f(x) = (2x^2 - 3x - 5)^2$
32. $h(x) = (x^2 - x - 20)(x^2 - 4)(2x + 1)$
33. $f(x) = (x - 3)^2(x + 1)$
34. $h(x) = (x - 2)^3(2x + 3)^2$

[4–4] Graph the following rational functions. State all asymptotes and intercepts.

35. $f(x) = \dfrac{2}{(x - 3)^3}$ **36.** $h(x) = \dfrac{-3}{(x + 5)^2}$

37. $f(x) = \dfrac{3}{x^2 - 4x - 45}$ **38.** $h(x) = \dfrac{2x}{2x^2 - 7x + 5}$

39. $g(x) = \dfrac{x^3}{(x - 1)(x^2 - 9)}$ **40.** $g(x) = \dfrac{x^2 - x - 6}{x^2 + 3}$

41. $f(x) = \dfrac{x^2 - x - 6}{x - 3}$

[4–5] Find expressions for $f + g$, $f - g$, $f \cdot g$, f/g, and $(f \circ g)(x)$ and $(g \circ f)(x)$ for the given functions f and g.

42. $f(x) = 3 - \frac{1}{2}x$; $g(x) = \frac{1}{2}x - 3$
43. $f(x) = x^4 - 1$; $g(x) = \sqrt{8 - x}$
44. $f(x) = \dfrac{x - 3}{2x}$; $g(x) = \dfrac{x}{2x - 1}$
45. $f(x) = -2x$; $g(x) = 3x$
46. $f(x) = x$; $g(x) = -3$

Find the inverse function of the given function. All the functions are one to one.

47. $g(x) = \dfrac{2x - 5}{4}$ **48.** $h(x) = \dfrac{x - 4}{2x + 1}$
49. $g(x) = x^2 + 8, x \geq 0$ **50.** $g(x) = 8x^3 - 27$
51. $g(x) = \sqrt[3]{1 - x} - 3$
52. $f(x) = x^2 - 7x + 6, x \geq 3\frac{1}{2}$
53. A solar engineer found that a solar hot water heating installation could heat 40 gallons per day in January and 200 in August. Create a linear function that models this situation of capacity c, as a function of time in months t, and use this function to predict the number of gallons of hot water heating capacity that the system might have in June.

[4–6] Decompose the following rational functions into partial fractions.

54. $\dfrac{8x + 11}{2x^2 - 5x - 3}$ **55.** $\dfrac{13x^2 - 52x + 32}{(x - 3)^2(2x + 1)}$

56. $\dfrac{2x^2 - 5x + 8}{(x - 2)(x^2 - x + 4)}$ **57.** $\dfrac{-2x^3 + 18x^2 - 53x + 62}{(x + 2)(x - 2)^3}$

Chapter 4 test

Graph the following parabolas. Compute the intercepts and vertex.

1. $y = x^2 + 5x - 14$ **2.** $y = -2x^2 + 8$
3. $y = 3x^2 + 5x - 2$ **4.** $y = -x^2 + 4x$

Solve the following two problems by creating an appropriate second-degree equation and finding the vertex.

5. A homeowner has 50 feet of fencing to fence off a rectangular area behind the home. The home will serve as one boundary (the fence is only necessary for three of the four sides). What are the dimensions of the maximum area which can be fenced off, and what is the area?

6. If an object is thrown vertically into the air with an initial velocity of v_0 ft/s, then its distance above the ground s, for time t, is given by $s = v_0t - 16t^2$. Suppose an object is thrown upward with initial velocity 48 ft/s; find how high the object will go and when it will return to the ground.

7. Graph the function $g(x) = \begin{cases} -\frac{1}{2}x - \frac{5}{2}, & x < -1 \\ x^2 + 2x - 1, & x \geq -1 \end{cases}$.

List all possible rational zeros for the following polynomials.

8. $x^4 - 3x^2 + 8$ **9.** $4x^3 - 4x^2 + 2x - 12$

10. Use synthetic division to (a) divide the polynomial by the divisor indicated and (b) to evaluate the function at the value indicated.

$$f(x) = 3x^4 - 2x^3 - 30x^2 - 20$$

a. $x + 3$ **b.** $f(-3)$

For each function in problems 11–13:

a. Use Descartes' rule of signs to find the number of possible real zeros.
b. List all possible rational zeros of each polynomial.
c. Find all rational zeros; state the multiplicity when greater than one.
d. Factor each as much as possible over the real numbers.
e. If there are any possible irrational zeros state the least positive integer upper bound and the greatest negative integer lower bound for these zeros.

11. $f(x) = 4x^3 - 4x^2 - x + 1$
12. $g(x) = x^4 + 3x^3 + 4x^2 + 3x + 1$
13. $h(x) = 3x^5 - 5x^4 - 23x^3 + 53x^2 - 16x - 12$

Graph the following polynomial functions. Compute all intercepts.

14. $f(x) = x^3 - 6x^2 + 3x + 10$
15. $g(x) = x^3 + 3x^2 - 9x - 27$
16. $f(x) = (x^2 - 1)^2$
17. $h(x) = (x^2 - 3x - 10)(x^2 - 3x + 10)$

Graph the following rational functions. Compute all intercepts and state all asymptotes.

18. $f(x) = \dfrac{2}{(x + 1)^2}$ **19.** $f(x) = \dfrac{-1}{(x - 2)^3}$

20. $f(x) = \dfrac{1}{x^2 + 2x - 24}$ **21.** $f(x) = \dfrac{-x}{x^2 - 4}$

22. $g(x) = \dfrac{x^2 - 7x + 12}{x^2 + 1}$ **23.** $f(x) = \dfrac{x^2 - x - 12}{x - 5}$

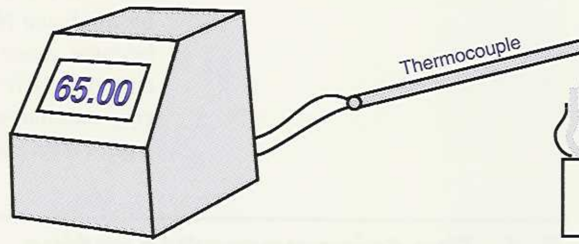
Find expressions for $f + g$, $f - g$, $f \cdot g$, f/g, and $(f \circ g)(x)$ and $(g \circ f)(x)$ for the given functions f and g.

24. $f(x) = 2x + 5$; $g(x) = x^2 - 2x - 4$
25. $f(x) = x^4 - 2$; $g(x) = 2\sqrt{x + 1}$
26. $f(x) = \dfrac{x + 2}{x}$; $g(x) = \dfrac{x}{2x - 1}$

Find the inverse function of the given function. All the functions are one to one.

27. $g(x) = 5x + 4$ **28.** $f(x) = \dfrac{5x + 1}{4x}$

29. $g(x) = x^2 - 4$, $x \geq 0$
30. $f(x) = x^2 - x - 6$, $x \leq \frac{1}{2}$
31. A thermocouple is an electronic device that can be used to measure temperature; its voltage output depends on (is a function of) temperature. A technician measured the output of a thermocouple to be 60 millivolts (mV) at 50° F and 80 mV at 100° F. Find a linear function that fits this data and use it to predict what the temperature is when the output of the thermocouple is 65 mV.



Decompose the following rational functions into partial fractions.

32. $\dfrac{5x^2 + 3x + 1}{(x + 1)^2(x - 2)}$ **33.** $\dfrac{4x^2 + 4x + 10}{(x - 1)(x^2 + x + 4)}$

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